Composite-Particle Interaction

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Abstract

A theoretical technique for description of composite nuclear particle interaction, in particular, the resonating group model and the orthogonality conditions model, is demonstrated. The discussion is focused on an algebraic version of the orthogonality conditions model proposed by the authors. In the framework of the developed formalism, it is possible to take into account among others the exchange terms of the kinetic energy operator precisely. Thus an approximation which is close to the original resonating group model, is built. Both a direct algebraic approach and a method based on the solution of integro-differential Schrödinger equation containing nonlocal terms related to forbidden and semi-forbidden states, are proposed as computational schemes. This equation turns out to be preferable in studies of narrow resonances. It is demonstrated that a decay width of a system to two-heavy-fragment channel is strongly affected by the nonlocal terms.

Keywords: *Clustering; Pauli principle; nucleus-nucleus interaction; resonance states*

1 Introduction

Properties of interaction of composite nuclear particles, i. e. the particles consisting of some identical fermion constituents, contrast dramatically with the properties of interaction of structureless ones. The principal origin of this contrast is the Pauli exclusion principle. As a consequence of the identity of the fermions composing two (or more) fragments in a realistic approach to the interaction of composite particles, it is necessary to take into account an antisymmetry of the wave function of the system as a whole and thus to consider internal multi-nucleon structures of the fragments. As a result, some eigenfunctions of the Hamiltonian describing the composite-particle interaction may vanish after the action of the antisymmetrization operator (forbidden states) or be renormalized by this operator (antisymmetrizer).

If the internal states of the interacting composite particles are fixed, the resonating group model (RGM) proposed in Refs. [1, 2] allows one to reduce the problem of description of their interaction to a two-body one (we do not consider systems of three or more fragments below). However the resulting two-body equation turns out to be not a Schrödinger-type one because it contains exchange integral kernels in all its terms. Methods of reduction of the RGM equation to the Schrödinger-type equation with a Hermitian Hamiltonian, are known (see, for example, Ref. [3]). However, even after these rearrangements, the RGM is an overcomplicated approach which looks still far from a habituated scheme of description of two-body interactions. Any pair of the composite particles should be described individually. In fact, the RGM remains a many-nucleon but not a two-body technique. Another problem of RGM is its inflexibility. As a consequence of incompleteness of the space of solutions inherent for the model, experimental observables are not well-reproduced within the RGM sometimes.

A goal of a lot of papers published after the original ones is to construct a method which, on the one hand, allows one to account for the property of fermion identity in

a system of two composite particles (and thus for the Pauli exclusion principle and various exchange effects caused by this identity) and, on the other hand, is relatively simple and flexible. The basic step on this way was made in the paper [4]. An approximation of the RGM which makes it more or less similar to an ordinary two-body approach is proposed there. Various alternatives of this approach have been developed up to now. These alternatives differ by methods of manipulation with the exchange terms. They are known under a unified name of orthogonality conditions model (OCM).

An algebraic version (AV) of OCM was proposed in Ref. [5]. It was built by analogy with the AV RGM proposed in Refs. [6, 7, 8]. Due to a unique potentiality of the algebraic formalism, it was possible to express precisely the exchange terms originated by the kinetic energy operator through eigenvalues of the so-called "norm" kernel of RGM and thus to construct a simple approximation which is very close to the original RGM.

In the present paper we demonstrate the lines of development of the method presented in Ref. [5]. In particular, an integro-differential equation of the Schrödinger type with a Hermitian Hamiltonian containing nonlocal terms related to the forbidden by the Pauli principle and semi-forbidden states, is obtained. This "comeback" to the methods of continuous mathematics turns out to be convenient in description of widths of narrow resonances decaying through cluster-cluster channels.

A number of characteristic examples including an interaction of light clusters, a nucleon-nucleus interaction, and an interaction of heavy ions, are considered. It is demonstrated by means of AV OCM that the values of the decay widths are strongly affected by the exchange effects in case of heavy-ion interactions. A pair of heavy magic fragments $^{16}\mathrm{O} + {}^{16}\mathrm{O}$ is considered as an example.

2 Composite-particle interaction in the framework of RGM and OCM. One-channel problem

2.1 RGM and OCM. Conventional formalism

Let as consider a traditional method of the description of composite particle interaction in the framework of RGM and its approximations used in OCM. Here we restrict this consideration to a one-channel problem for the sake of brevity.

The wave function of RGM [1, 2] is chosen in the form:

$$\Psi_{A_1+A_2} = \hat{A}\left\{\Psi_{A_1}\Psi_{A_2}\Phi\left(\rho\right)\right\}, \tag{1}$$

$$\hat{A} = \left(\begin{array}{c} A \\ A_1 \end{array}\right)^{-1/2} \left(1 + \sum_p \left(-1\right)^p \hat{P}\right), \tag{2}$$

where the sum is over all permutations $\hat{P}$ of A ($A = A_1 + A_2$) nucleons, p is the parity of the permutation and $\Phi\left(\rho\right)$ is a probe wave function. Inserting (1) into the A-fermion equation

$$\hat{H}_A\Psi_A = E\Psi_A, \tag{3}$$

$$\hat{H}_A = \hat{T} + \hat{V}, \quad \hat{T} = \sum_{i=1}^{A} \frac{\hat{\mathbf{p}}_i^2}{2m}, \tag{4}$$

$$\hat{V} = \sum_{i<j=1}^{A} V\left(\mathbf{r}_i - \mathbf{r}_j\right), \tag{5}$$

one can obtain a two-body equation:

$$\left(\hat{T}_\rho + \hat{V}_\rho - E'\hat{N}_\rho\right)\Phi\left(\rho\right) = 0, \tag{6}$$

where

$$E' = E - E_1 - E_2, \tag{7}$$

and the normalization condition

$$\left\langle \hat{N}_\rho^{1/2}\Phi(\rho) \,\middle|\, \hat{N}_\rho^{1/2}\Phi(\rho) \right\rangle = (1,\ \delta(E-E'),\ \delta(k-k'),\ \text{etc.}) \tag{8}$$

for bound and continuous states respectively. For a fixed value of the angular momentum of the relative motion l, the integral operators can be presented as

$$\begin{pmatrix} \hat{N}_{\rho,l} \\ \hat{T}_{\rho,l} \\ \hat{V}_{\rho,l} \end{pmatrix} \varphi_l(\rho) \equiv \int \begin{pmatrix} N_l(\rho',\rho) \\ T_l(\rho',\rho) \\ V_l(\rho',\rho) \end{pmatrix} \varphi_l(\rho')\,\rho'^2 d\rho', \tag{9}$$

$$\begin{pmatrix} N_l(\rho',\rho'') \\ T_l(\rho',\rho'') \\ V_l(\rho',\rho'') \end{pmatrix}$$

$$= \left\langle \hat{A}\left\{\Psi_{A_1}\Psi_{A_2}\frac{1}{\rho^2}\delta(\rho-\rho')\,Y_{lm}(\Omega_\rho)\right\} \middle| \begin{pmatrix} \hat{1} \\ \hat{T} \\ \hat{V} \end{pmatrix} \middle| \hat{A}\left\{\Psi_{A_1}\Psi_{A_2}\frac{1}{\rho^2}\delta(\rho-\rho'')\,Y_{lm}(\Omega_\rho)\right\}\right\rangle. \tag{10}$$

Thus the discussed two-body equation turns out to be an integro-differential equation of the form which differs from the Schrödinger one.

The equation (6) can be transformed to a Schrödinger-like form by action of the operator $\hat{N}_{\rho,l}^{-1}$:

$$\left(\hat{N}_{\rho,l}^{-1}\hat{T}_{\rho,l} + \hat{N}_{\rho,l}^{-1}\hat{V}_{\rho,l} - E'\right)\varphi_l(\rho) = 0, \tag{11}$$

but the resulting Hamiltonian turns out to be a non-Hermitian one. Introducing a new wave function

$$\phi_l(\rho) = \hat{N}_{\rho,l}^{1/2}\,\varphi_l(\rho), \tag{12}$$

one can obtain a Schrödinger-like equation with a Hermitian Hamiltonian

$$\left(\hat{N}_{\rho,l}^{-1/2}\hat{T}_\rho\hat{N}_{\rho,l}^{-1/2} + \hat{N}_{\rho,l}^{-1/2}\hat{V}_{\rho,l}\hat{N}_{\rho,l}^{-1/2} - E'\right)\phi_l(\rho) = 0 \tag{13}$$

and usual normalization conditions:

$$\langle \phi_{E,l}(\rho)\,|\,\phi_{E',l}(\rho)\rangle = 1 \tag{14}$$

for states of discrete spectrum, and

$$\langle \phi_{E,l}(\rho)\,|\,\phi_{E',l}(\rho)\rangle = \delta(E-E') \tag{15}$$

(or similar) for states in continuum.

The basic approximation of the original OCM [4] is

$$V(\rho',\rho'') = V(\rho')\,\delta(\rho'-\rho''). \tag{16}$$

If the forbidden components are extracted from the function sought, then the initial equation becomes

$$\left(\hat{T}_\rho + \hat{V}(\rho) - E'\right)\tilde{\Phi}(\rho) = 0, \tag{17}$$

where $\hat{V}(\rho)$ is a direct (double folding) potential. Usually the exchange terms are neglected in the kinetic energy operator. In this approximation,

$$\hat{T}_\rho = \frac{\hat{\mathbf{p}}_\rho^2}{2\mu}. \tag{18}$$

In some cases, a quality of description of data is not high enough in the initial version of OCM. Other versions of OCM (presented, e. g., in Refs. [9, 10]) utilize a phenomenological local potential in contrast to the direct one and explore two alternative ways to take into account the Pauli exclusion principle:

1. A two-body model with forbidden states which are eigenstates of the Hamiltonian. This version is rather simple because the redundant states are easily excluded due to their orthogonality to the others in this case.

2. A two-body model with forbidden states which are eigenstates of the norm kernel $\hat{N}_\rho$.

A possibility of adjusting the two-body local potential makes the approaches more flexible and improves the quality of the results. At the same time, the resulting OCM (notably with the forbidden eigenstates of the Hamiltonian) occurs to be an approximation of the RGM falling far from the original model. Therefore more accurate approximations are of interest for the theory of composite-particle interaction.

2.2 Algebraic version of RGM and the developed version of OCM.

Within RGM, the functions Ψ_{A_1} and Ψ_{A_2} are most often considered as the ground state (i. e. the lowest compatible with the Pauli exclusion principle) oscillator wave functions with the same parameter $\hbar\omega$. In the algebraic version of RGM, the relative motion function is sought in the form of expansion

$$\varphi_l(\rho) = \sum_n C_{nl}\,\phi_{nl}(\rho) \tag{19}$$

in oscillator basis functions $\phi_{nl}(\rho)$ (also characterized by the same parameter $\hbar\omega$). Under these conditions, the wave functions $\phi_{nl}(\rho)$ are eigenfunctions of the norm kernel:

$$\hat{N}_{\rho,l}\,\phi_{nl}(\rho) = \varepsilon_n\,\phi_{nl}(\rho). \tag{20}$$

The eigenvalues ε_n are equal to zero for the forbidden states and tend to unity as $n \to \infty$. Semi-forbidden states are defined as states with ε_n considerably lower then the unity. There are rare cases for which the eigenvalues are higher than the unity. A mathematical formalism described below also allows one to include these states in a similar way as semi-forbidden states, thus we do not discuss this case separately.

A fundamental advantage of AV OCM is a possibility to apply the following relation presented in Ref. [11]:

$$\left\langle \phi_{nl} \left| \hat{N}_{\rho,l}^{-1/2} \hat{T}_\rho \hat{N}_{\rho,l}^{-1/2} \right| \phi_{n'l} \right\rangle = \sqrt{\frac{\varepsilon_{n_<}}{\varepsilon_{n_>}}}\, T_{nn'}, \tag{21}$$

where $n_< = \min(n, n')$, $n_> = \max(n, n')$ and $T_{nn'}$ is the matrix element of the ordinary two-body kinetic energy operator between the oscillator functions. Due to this, the set of the AV RGM equations looks as follows:

$$\sum_{n'} \left[\sqrt{\frac{\varepsilon_{n_<}}{\varepsilon_{n_>}}}\, T_{nn'} + \left(\hat{N}_{\rho,l}^{-1/2} \hat{V}_{\rho,l} \hat{N}_{\rho,l}^{-1/2} \right)_{nn'} - E' \delta_{nn'} \right] C_{n'l} = 0; \quad n \geq n_{\min}. \tag{22}$$

So, the A-fermion exchange properties of the kinetic energy operator are precisely determined by the eigenvalues of the norm kernel.

The approach present here allows one, first, to take into consideration the exchange terms of the kinetic energy completely and, second, to use an alternative (well-grounded microscopically) way of exclusion of the forbidden states.

It should be noted that due to the equality (21), the potential energy term $\hat{V}_{\rho,l}$ contained in Eqs. (13) and (22) remains the only term of the Hamiltonian in AV RGM equations that includes the fermion exchange operators in the explicit form. It is just the term which is responsible for turning out RGM into a non-universal and overcomplicated model. The idea is to consider the term $\hat{N}_{\rho,l}^{-1/2}\hat{V}_{\rho,l}\hat{N}_{\rho,l}^{-1/2}$ phenomenologically, approximating it by a local potential $\hat{V}_{cl}(\rho)$. As a result, we obtain the equation

$$\left(\hat{\hat{T}}_\rho + \hat{V}_{cl}\left(\rho\right) - E'\right)\tilde{\Phi}\left(\rho\right) = 0. \tag{23}$$

It should be noted that the choice of another scheme considering as a local potential the term $\hat{V}_{\rho,l}$ instead of $\hat{N}_{\rho,l}^{-1/2}\hat{V}_{\rho,l}\hat{N}_{\rho,l}^{-1/2}$, is also possible and does not present any additional problem. The latter choice looks less reasonable because the semi-forbidden states with small values of ε_n (if such values exist in a particular example) may cause an instability in the fitting procedure of the parameters of the local potential in this case.

The approach is called AV OCM independently of methods (which may be algebraic or that of "continuous" mathematics) applied to solve it.

In the algebraic versions of the canonic two-body problem, RGM as well as OCM presented here, the expansion coefficients C_{nl} of Eq. (19) satisfy an infinite set of linear equations

$$\sum_{n=0}^{\infty}\left(\langle\phi_{n'lm}|\,\hat{H}\,|\phi_{nlm}\rangle - E\delta_{n'n}\right)C_{nl} = 0, \quad n' = 0,\ 1, \ldots, \tag{24}$$

which follows from the respective Schrödinger equation. For ordinary bound states, the eigenvalue problem,

$$\det\left\|\hat{H} - E\hat{I}\right\| = 0, \tag{25}$$

is solved on the truncated basis with $n \le n_{\max}$. Here truncation means the boundary condition $C_{nl} = 0, \quad n > n_{\max}$ in the oscillator representation. For states of continuous spectrum (including rather broad near-barrier resonances), the convergence of the functional series (19) is not uniform; therefore the so-called J-matrix method [12] is applied. The expansion coefficients decrease rather slowly with n, and their asymptotic behavior should be introduced in the set of equations:

$$\sum_{n=0}^{N-1}\left(\langle\phi_{n'lm}|\,\hat{H}\,|\phi_{nlm}\rangle - E\delta_{n'n}\right)C_{nl} = -\sum_{n=N}^{\infty}\langle\phi_{n'lm}|\,\hat{H}\,|\phi_{nlm}\rangle\,C_{nl}^{(as)}, \quad n' = 0,\ 1, \ldots \tag{26}$$

The papers [6, 7, 8] were the first works in which the discussed method was applied to solve the RGM equations.

A high-precision form of the asymptotic coefficients was obtained in Refs. [13, 14, 15]. In particular, for the wave function asymptotically behaving as an outgoing Coulomb wave, the expansion coefficient has the following form:

$$C_{nl}^{(as)} = \frac{1}{\sqrt{kr_0\varsigma_n}}\left\{G_l\left(\eta, kr_0\varsigma_n\right) + iF_l\left(\eta, kr_0\varsigma_n\right) - \frac{k^3r_0^3}{6\varsigma_n}\left[G_l'\left(\eta, kr_0\varsigma_n\right) + iF_l'\left(\eta, kr_0\varsigma_n\right)\right]\right\}, \quad n \to \infty, \tag{27}$$

where $r_0 = \sqrt{\hbar/\mu\omega}$ is the oscillator radius, $\varsigma_n = \sqrt{2n+3}$, the prime denotes the derivatives of the Coulomb wave functions with respect to the second argument. The first term in the figure brackets of Eq. (27) provides a rather good approximation in most cases.

A related approach may be also useful for calculations of near-threshold bound states.

The presence of the forbidden states in Eq. (20) restricts the set of equations (26) and the sums in the left-hand side of them by the conditions $n, n' \geq n_{\min}$. Semi-forbidden states are taken into account by means of renormalization of the kinetic energy matrix. The explicit form of the of the kinetic energy matrix elements (21) is applied for that. The matrix of the renormalized kinetic energy operator (21) retains a tridiagonal form in the oscillator basis as the initial one.

The method developed here is applicable to the calculations of phase shifts and cross-section of composite particle scattering including calculations in the framework of the optical model, near-threshold bound cluster-nucleus states, resonance states of various cluster-cluster pairs excluding too narrow resonances, amplitudes of entrance and exit channels of various reactions.

It should be noted that the one-channel formalism developed here is valid in the case when both clusters are SU(3)-scalars (it is true for magic and light clusters) and, in addition, one of them is an SU(4)-scalar. Otherwise a channel coupling appears due to the antisymmetrization. As a result, some modifications of the technique are required. The quality of the one-channel approximation in this multi-channel problem depends on the dynamics of the channel under investigation.

2.3 AV OCM. Equivalent integro-differential equation.

In some cases it is hard to explore the direct algebraic approach presented above because the asymptotic behavior of the expansion coefficients is achieved at too large distances, and a huge basis in Eq. (19) is required. It is the case of a narrow resonance in a system decaying through a two charged composite-particle channel. In this situation, it occurs more convenient to apply methods of "continuous" mathematics. To do this, a number of separable terms related to forbidden and semi-forbidden states are introduced into the Hamiltonian. The idea of this rearrangement is that the initial cluster Hamiltonian matrix elements $H_{nn';l}$ between the states at least one of which is forbidden, are cancelled by the corresponding matrix elements of separable terms. The kinetic energy matrix elements in the Hamiltonian are renormalized according to the formula (21) to account for the presence of semi-forbidden states. The additional potential term denoted as $\hat{V}_l^{sep}$, takes the form:

$$\hat{V}_l^{sep} = -\sum_{n,n'=0}^{n_0} |nl\rangle\, H_{nn';l}\, \langle n'l| - \sum_{n=0}^{n_0}\sum_{n'=n_0+1}^{\infty} (|nl\rangle\, H_{nn';l}\, \langle n'l| + |n'l\rangle\, H_{n'n;l}\, \langle nl|)$$
$$+ \sum_{n=n_0+1}^{\infty} \left(\sqrt{\frac{\varepsilon_n}{\varepsilon_{n+1}}} - 1\right) (|nl\rangle\, T_{n,n+1;l}\, \langle n+1,l| + |n+1,l\rangle\, T_{n+1,n;l}\, \langle nl|), \quad (28)$$

where

$$\hat{H} = \hat{T}_\rho + \hat{V}_{cl}(\rho). \quad (29)$$

Usually $\varepsilon_n/\varepsilon_{n+1}$ tends to unity rather rapidly as n increases (for example, $\varepsilon_{n=80} = 0.999$ for the system $^{16}\mathrm{O} + {}^{16}\mathrm{O}$), therefore the sum in Eq. (28) can be truncated by a relatively small value of n_0.

Here we demonstrate an appropriate method to solve the Schrödinger equation

$$\left(\frac{d^2}{dr^2} + k^2 - 2V_{cl;l}(\rho)\right) \chi_l(\rho) = 2\hat{V}_l^{sep}\chi_l(\rho) \quad (30)$$

with the additional separable terms. The solution of this equation $\chi_l(\rho)$ must behave asymptotically as $G_l(\eta, k\rho) + iF_l(\eta, k\rho)$ at large distances. $G_l(\eta, k\rho)$ exceeds significantly $F_l(\eta, k\rho)$ in the under-barrier domain far enough from the external turning

point. Thus, according to Ref. [16], it is enough to satisfy the matching condition in this region with the function $G_l(\eta, k\rho)$ only. Consequently, the solution $\chi_l(\rho)$ should satisfy the following boundary conditions:

(i) it is regular at the origin ($\rho = 0$);

(ii) it behaves as $G_l(\eta, k\rho)$ under the barrier beyond the radius of the strong interaction.

To find the solution of Eq. (30), let us consider the equation

$$\left(\frac{d^2}{dr^2} + k^2 - 2V_{cl;l}(\rho)\right)\chi_l(\rho) = 0 \tag{31}$$

with a local potential $V_{cl;l}(\rho)$ including the centrifugal part, and introduce its solutions $\chi_{1;l}(\rho)$ which satisfies the condition (i) and $\chi_{2;l}(\rho)$ which satisfies (ii). The corresponding Green's function takes the form

$$G(\rho, \rho') = \frac{\chi_{1;l}(\rho_<)\,\chi_{2;l}(\rho_>)}{W}, \tag{32}$$

where $\rho_< = \min(\rho, \rho')$, $\rho_> = \max(\rho, \rho')$, and the Wronskian W is written as follows:

$$W = \chi_{1;l}(\rho)\,\frac{d\chi_{2;l}(\rho)}{d\rho} - \frac{d\chi_{1;l}(\rho)}{d\rho}\,\chi_{2;l}(\rho). \tag{33}$$

This Green's function allows one to deduce the homogeneous integral equation

$$\chi_l(\rho) = -2\int_0^\infty G(\rho, \rho')\left[\hat{V}_l^{sep}\chi_l\right](\rho')\,d\rho' \tag{34}$$

for the resonance solution $\chi_l(\rho)$. Here $\left[\hat{V}_l^{sep}\chi_l\right](\rho')$ means the function of ρ' which is the result of action of the operator $\hat{V}_l^{sep}$ on the function $\chi_l(\rho)$. The homogeneous equation (34) has solutions only for unique resonance energy values. Substituting $\hat{V}_l^{sep}$ in Eq. (34) by its explicit expression (28), we obtain:

$$\begin{aligned}
\chi_l(\rho) = 2\sum_{n=0}^{n_0}\int_0^\infty G(\rho, \rho')\,\phi_{nl}(\rho')\,\langle\phi_{nl}|\,V_{cl;l}\,|\chi_l\rangle\,d\rho' \\
+ 2\left[\int_0^\infty d\rho'\,G(\rho, \rho')\,\phi_{n_0 l}(\rho')\,T_{n_0, n_0+1;l}\right. \\
+ \int_0^\infty d\rho'\,G(\rho, \rho')\,\phi_{n_0+2,l}(\rho')\left(1 - \sqrt{\frac{\varepsilon_{n_0+1}}{\varepsilon_{n_0+2}}}\right)T_{n_0+1, n_0+2;l}\Bigg]\,\langle\phi_{n_0+1,l}\mid\chi_l\rangle \\
+ 2\sum_{n=n_0+2}^{n_{\max}-1}\left[\int_0^\infty d\rho'\,G(\rho, \rho')\,\phi_{n-1,l}(\rho')\left(1 - \sqrt{\frac{\varepsilon_{n-1}}{\varepsilon_n}}\right)T_{n-1,n;l}\right. \\
\left. + \int_0^\infty d\rho'\,G(\rho, \rho')\,\phi_{n+1,l}(\rho')\left(1 - \sqrt{\frac{\varepsilon_n}{\varepsilon_{n+1}}}\right)T_{n,n+1;l}\right]\,\langle\phi_{nl}\mid\chi_l\rangle \\
+ 2\int_0^\infty d\rho'\,G(\rho, \rho')\,\phi_{n_{\max}-1,l}(\rho')\left(1 - \sqrt{\frac{\varepsilon_{n_{\max}-1}}{\varepsilon_{n_{\max}}}}\right)T_{n_{\max}-1, n_{\max};l}\,\langle\phi_{n_{\max},l}\mid\chi_l\rangle,
\end{aligned} \tag{35}$$

where n_{max} means the maximum value of the radial quantum number of the truncated oscillator basis. A simple form of the first term is caused by the completeness of the oscillator basis allowing one to calculate the infinite sum over n' in the expression (28) explicitly.

There is an opportunity to treat Eq. (35) in the following way. Multiplying it by $\langle\phi_{nl}|$ and $\langle\phi_{nl}|\,V_{cl;l}$ one can obtain a set of homogeneous algebraic equations for the unknown coefficients $\langle\phi_{nl}|\chi_l\rangle$ and $\langle\phi_{nl}|\,V_{cl;l}\,|\chi_l\rangle$. The condition of solvability (zero value of the determinant) determines the value of E_{res}, after that the coefficients $\langle\phi_{nl}|\chi_l\rangle$ and $\langle\phi_{nl}|\,V_{cl;l}\,|\chi_l\rangle$ can be calculated. This procedure determines the function $\chi_l(\rho)$ and thus the width of the resonance.

However such a method of numerical calculations of widths turns out to be unstable at least for narrow resonances in systems possessing a number of semi-forbidden states with eigenvalues of the norm kernel essentially different from the unity. In particular, a very high accuracy (ten significant digits for the 2 MeV resonance in the $^{16}\mathrm{O} + {}^{16}\mathrm{O}$ system) of the E_{res} value is required to calculate the width reliably.

A way to overcome this difficulty looks as follows. Consider the above mentioned function obtained via the direct algebraic approach:

$$\tilde{\chi}_l(\rho) = \sum_{n=n_0+1}^{n_{\max}} C_{nl}\ \phi_{nl}(\rho). \tag{36}$$

This function is a partial sum of the oscillator expansion of $\chi_l(\rho)$. It reproduces precisely the behavior of the wave function $\chi_l(\rho)$ in the interior domain. It is just what is needed to calculate the values of $\langle\phi_{nl}|\chi_l\rangle$ and $\langle\phi_{nl}|\,V_{cl;l}\,|\chi_l\rangle$ due to a rapid decrease of the functions $\phi_{nl}(\rho)$, $V_{cl;l}(\rho)$ and $\chi_l(\rho)$ with ρ. Thus, substituting $\chi_l(\rho)$ in the right-hand side of the basic equation (35) by $\tilde{\chi}_l(\rho)$, one can obtain the solution for all values of ρ including the asymptotic region. Numerical calculations by means of the proposed method occur to be significantly more stable.

The width of a narrow resonance can be obtained from the solution $\chi_l(\rho)$ by means of the approach presented in the monograph [16]. According to this approach, the following asymptotic relation

$$\chi_l(\rho) \simeq \sqrt{\frac{\Gamma k}{2E_{res}}}\, G_l(\eta, k\rho) \tag{37}$$

is valid in the case $G_l(\eta, k\rho) \gg F_l(\eta, k\rho)$ in the under-barrier region far enough from the external turning point. The normalization condition

$$\int_0^R \chi_l^2(\rho)\ d\rho = 1 \tag{38}$$

in the interior region is implied. The described above method of calculation of the resonance wave function with the aid of Eq. (35), allows one to apply directly the formula (37) to determining the decay width Γ.

3 AV OCM and exchange effects in decay processes

3.1 Alpha-decay of 91.84 keV 0^+ resonance in ^{8}Be

To analyze the interrelation between various approaches based on OCM and RGM models, a canonical object of the physics of clustering, the 91.84 keV 0^+ resonance in the ^{8}Be nucleus, was studied in Ref. [17].

The width of the ground state of ^{8}Be nucleus which is a low-laying resonance of the α-α system (the experimental value of the width $\Gamma = 6.8$ eV), was calculated

in various versions of two-body, OCM and RGM dynamics. The proper resonance energy $E = 91.84$ keV was achieved, if necessary, by fitting the depth of the potential well. The following results were obtained.

For an illustration of the results obtained in two-body models with forbidden eigenstates of the Hamiltonian, we consider the model with the Buck potential proposed in Ref. [18] which has the form

$$V(\rho) = V_0 \exp(-b\rho^2) + V_{Coul}, \tag{39}$$

where

$$V_{\text{Coul}}(\rho) = \left(Z_1 Z_2 e^2/\rho\right) \operatorname{erf}(\rho/d), \tag{40}$$

and parameters $V_0 = 122.6$ MeV, $b = 0.22$ fm^{-2} and $d = 1.33$ fm. The fitting is not required in this case. The value of the resonance energy $E = 91.10$ keV is reproduced in this dynamics. The value of the width coincides with the experimental one ($\Gamma = 6.8$ eV).

The same two-body model with forbidden states which are eigenstates of the kernel $\hat{N}_\rho$, is used for an illustration of this type of two-body models. The procedure of the resonance energy fit by variation of the potential well depth results in the value $V_0 = 116.9$ MeV. The values deduced for this version are: $E = 91.84$ keV, $\Gamma = 5.8$ eV. Thus the properties of the discussed channel obtained in these two versions of the dynamics, are close enough.

The straightforward RGM calculation using the Hasegava–Nagata NN potential (see, for example, Ref. [19]) with no fitting results in the values $E = 91.84$ keV, $\Gamma = 4.9$ eV.

At last, the OCM version with the RGM-projected kinetic energy operator (21) and forbidden states which are eigenstates of the kernel $\hat{N}_\rho$, is analyzed. The Gaussian form of the phenomenological potential with the width equal to the width of the Buck potential ($b = 0.22$ fm^{-2}), results in the values: $V_0 = 136.1$ MeV, $E = 91.84$ keV, $\Gamma = 4.7$ eV. Thus the direct inclusion of semi-forbidden states changes the local potential significantly and the results obtained in RGM are well-reproduced by AV OCM.

3.2 Asymptotic normalization coefficient for loosely bound state of the ^{17}F nucleus

To analyze weakly bound states in the framework of AV OCM, the closed channel $^{16}\text{O} + p$, $J^\pi = 1/2^+$ at the energy $E_p = -104.94$ keV was studied in Ref. [20]. This sub-threshold resonance is actively analyzed for astrophysical purposes [21]. The asymptotic behavior of the radial wave function in the two-body model is expressed through the Whittaker function:

$$\phi_l(\rho) \to D_l \, W_{-\eta, l+1/2}(2k\rho)/\rho, \tag{41}$$

where

$$\eta = Z_1 Z_2 e^2 \mu/\hbar^2 k \tag{42}$$

is the Coulomb parameter.

The asymptotic normalization coefficient D_l is the factor determining the amplitude of of the wave function as $\rho \to \infty$. In the algebraic version of OCM, the coefficients C_{nl} obtained as the solution of the set of equations (26) are compared with the asymptotic ones which, by analogy with the first approximation of the formula (27), take the form:

$$C_n^{(as)} = \sqrt{r_0}\, [4/(2n+3)]^{1/4}\, W_{-\eta, l+1/2}(2k\rho_n)\, D_l, \tag{43}$$

where $\rho_n = r_0 \varsigma_n$ is the turning point of the oscillator wave function $\phi_{nl}(\rho)$. The matching condition $C_n = C_n^{(as)}$ determines the coefficient D_l.

Table 1: The eigenvalues of the overlap kernel for $^{16}\mathrm{O} + N$.

n	0	2	4	6
ε_n	0	1.128906	1.001022	1.000006

Table 2: Depth of the nucleon-nucleus potential V_0 and asymptotic normalization coefficient D_l for the $^{16}\mathrm{O} + p$ system within TBM and AV OCM.

Alternative	V_0, MeV	D_l, fm$^{-1/2}$
TBM	49.24	83.33
OCM	47.61	94.18

The nucleon-nucleus potential is chosen in the form:

$$V(\rho) = -V_0 \left\{1 + \exp\left[(\rho - R_0)/a\right]\right\}^{-1} + V_c(\rho), \tag{44}$$

$$V_c(\rho) = \begin{cases} (4\alpha_e\hbar c/R_c)(3 - \rho^2/R_c^2), & \rho < R_c, \\ 8\alpha_e\hbar c/\rho, & \rho > R_c, \end{cases} \tag{45}$$

where $\alpha_e = e^2/\hbar c$, and parameters $R_0 = 3.29$ fm, $a = 0.65$ fm, $R_c = 3.48$ fm are used in the model calculations. A numerical solution of the two-body Schrödinger equation is used to test the accuracy of both two-body and the AV OCM variational calculations.

For the discussed channel, the eigenvalues of the norm kernel which are involved in the expression of the renormalized kinetic energy, can be calculated using the formula:

$$\varepsilon_n = 1 + (-1)^n(17n - 1)/16^n. \tag{46}$$

They are presented in Table 1. As is seen, only $\varepsilon_{n=2}$ differs significantly from the unity.

The values of the D_l coefficient are calculated both in the ordinary two-body model (TBM) and in AV OCM with the RGM-projected kinetic energy operator and forbidden states which are eigenstates of the kernel $\hat{N}_\rho$. The depth of the local nuclear potential (44) is varied to fit the proton binding energy of 104.94 keV just in the same manner as in the examples of the previous subsection. The results of these calculations are presented in Table 2.

As is seen from Table 2, the inclusion of the exchange terms increases the value of the asymptotic normalization coefficient by 10 % in comparison with the two-body one. Microscopic calculations of the asymptotic normalization coefficient in the framework of RGM using NN potentials from Refs. [22] and [23] result in the values of 91.15 fm$^{-1/2}$ and 86.20 fm$^{-1/2}$ respectively. The value of 85.65 fm$^{-1/2}$ was obtained in our RGM calculations with the Hasegava–Nagata NN potential [19].

3.3 Width of the lowest $^{16}\mathrm{O} + {}^{16}\mathrm{O}$ resonance state

To demonstrate the effect of forbidden and the semi-forbidden states on the decay widths of nuclear states in the case of emission of heavy clusters, let us consider the pair $^{16}\mathrm{O} + {}^{16}\mathrm{O}$ as an example. Three alternatives are studied:

(I) OCM with forbidden states considered as eigenstates of the two-body Hamiltonian,

(II) OCM with forbidden states considered as eigenstates of the norm kernel,

Table 3: Width of the lowest $^{16}O + {}^{16}O$ resonance state for three versions of the interaction (see the text).

Alternative	(I)	(II)	(III)
V_0, MeV	399.2	225.6	422.8
E_{res}, MeV	2.103	2.103	2.102
Γ, MeV	$0.59 \cdot 10^{-27}$	$0.53 \cdot 10^{-28}$	$0.64 \cdot 10^{-35}$

(III) OCM with forbidden states of the latter type and semi-forbidden states.

The local cluster-cluster potential from Ref. [24]

$$V_{cl}(\rho) = V_{Coul}(\rho) + \frac{V_0}{\left(1 + \exp\left[\left(\rho - R\right)/a\right]\right)^2} \tag{47}$$

is considered. The Coulomb part is chosen in the form of interaction potential of two uniformly charged spherical volumes. According to one of the versions of the model used in Ref. [24], there are 12 forbidden states (eigenstates of the Hamiltonian with the interaction (47)) and a narrow resonance state at the energy of $E = 2.103$ MeV in the partial wave with $l = 0$. This result is reproduced in our calculations realizing the alternative (I). For the alternatives (II) and (III), the depth V_0 of the local potential $V_{cl}(\rho)$ is varied to restore the resonance energy of the alternative (I). The values of the decay width for three versions of OCM are presented in Table 3. The resonance energy is presented to demonstrate the accuracy of its reproduction.

It is clear from the Table that the forbidden and notably the semi-forbidden states change drastically the decay width. If the forbidden states are considered as eigenstates of the norm kernel, the value of the width Γ becomes one order of magnitude smaller than the one obtained assuming these states to be the eigenstates of the two-body Hamiltonian. If, in addition, the semi-forbidden states are also considered, the value of the width Γ turns out to be eight orders of magnitude smaller. Thus a very pronounced exchange effect manifests itself in the properties of a resonance decaying through a channel with a long list of semi-forbidden states which differ significantly from the unity.

It should be noted that parameters of the $^{16}O + {}^{16}O$ channel (the penetrability of the barrier, the eigenvalues of the norm kernel for the semi-forbidden states, the Γ values) are more or less close to the ones typical for alpha-decays of heavy nuclei. Therefore one may expect similar exchange effects in the latter process.

4 Summary

In the present paper, results of the study of a new version of the orthogonality conditions model are demonstrated in details. The model allows one to take into account exchange effects originating from the norm and kinetic energy overlap kernels. Both continuous and pure algebraic formalisms of the model are developed. The former one is used for the calculation of decay widths of very narrow resonances. The pairs of particles $\alpha + \alpha$, $^{16}O + p$ and $^{16}O + {}^{16}O$ are given as examples. The results of the study demonstrate that:

1. Properties of interaction of composite particles are essentially different from the ones of structureless particles.

2. The basic cause of the differences is the exchange effects manifesting themselves via forbidden and semi-forbidden states.

3. Algebraic approaches are convenient tools for considering these effects.

4. The methods developed here for description of composite particle interaction are applicable to the calculations of:

   a) phase shifts and cross-sections of composite particle scattering including calculations in the framework of the optical model,

   b) near-threshold bound cluster-nucleus states,

   c) resonance states of various cluster-cluster pairs including very narrow resonances,

   d) amplitudes of entrance and exit channels of various reactions.

5. The formalism developed is valid in the case of both clusters being SU(3)-scalars and additionally one of them being an SU(4)-scalar.

6. The effect of semi-forbidden states is drastic in calculations of widths of narrow resonances in interaction of a heavy nucleus with alpha-particle or in interaction of two heavy clusters.

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