Infrared and Ultraviolet Cutoffs: Convergence Strategies for Harmonic Oscillator Basis Expansion Methods

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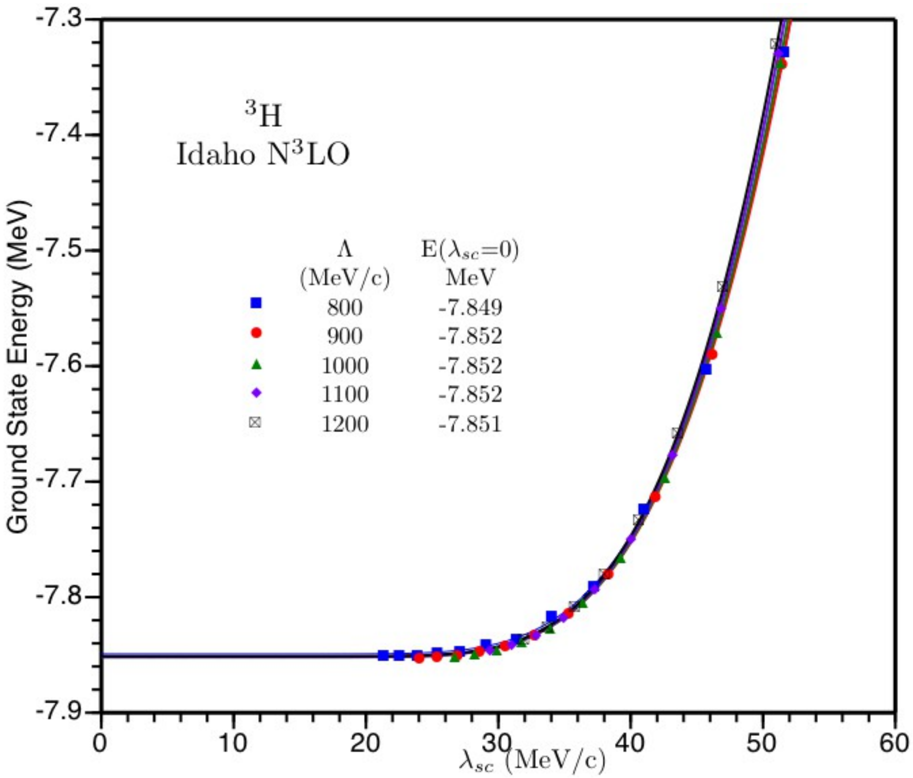
New extrapolation



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Archive:1205.3230, PRC 86, 054002 (2012)

Traditional extrapolation

Chiral NN interaction, λ = 2.5 fm-1
Extrapolation A'

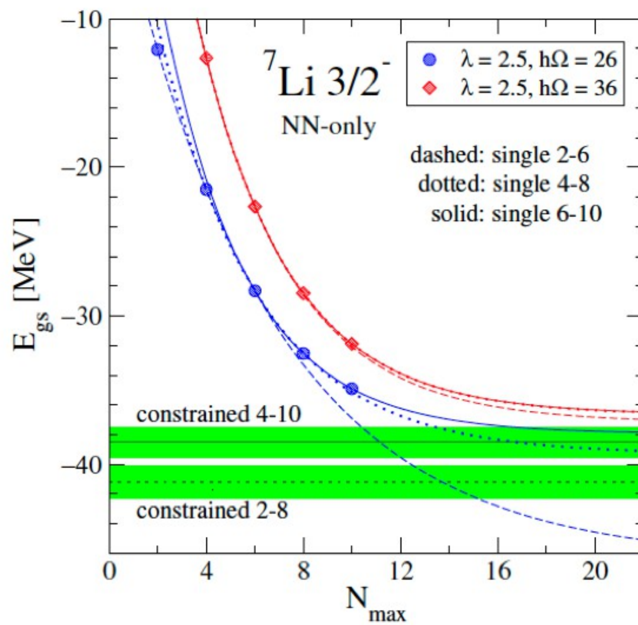


FIG. 14. (color online) Ground-state energy extrapolations of ^{7}Li as a function of N_{max} with an N^{3}LO NN interaction [21] evolved to $\lambda = 2.5\,\mathrm{fm}^{-1}$. The symbols are the calculated points. The curves show single extrapolations using Eq. (3) with $N_{max} = 2$–6 (dashed), 4–8 (dotted) and 6–10 (solid) at (blue) $\hbar\Omega = 26$ MeV which minimize the amount of extrapolation and at (red) $\hbar\Omega = 36$ MeV which minimize the numerical error estimate. The horizontal dotted and solid lines, with the band indicating the associated error bars, are the result from a constrained fit following the procedure of Ref. [1] for five $\hbar\Omega$ values from 22 to 30 MeV.

Note the inconsistency – what's happening?

E.D. Jurgenson, P. Maris, R.J. Furnstahl, P. Navratil, W.E. Ormand, J.P. Vary, submitted to PRC; arXiv: 1302:5473

The traditional shell-model calculation involves trial variational wave functions which are linear combinations of Slater determinants. Expanding the conf guration space merely serves to improve the trial wave function.

•J. M. Irvine, et al “Nuclear shell-model calculations and strong two-body correlations”, Ann. Phys. 102, 129 (1976).

early appearance of the term “no-core shell model” (NCSM)

Use the HO eigenfunctions as a basis of a finite linear expansion to make a straightforward variational calculation of the properties of light nuclei.

M. Moshinsky, “The Harmonic Oscillator in Modern Physics: from Atoms to Quarks” (Gordon and Breach, New York, 1969).

$$\Psi_T = \sum_{\nu} a_{\nu}^{(\mathcal{N})} h_{\nu}$$

where $a_{\nu}^{(\mathcal{N})}$ are the parameters to be varied and h_{ν} are many-body states based on a summation over products of HO functions.

Theorems based upon functional analysis established the asymptotic convergence rate of these calculations as a function of the counting number N which characterizes the size of the expansion basis (or model space)

•inverse power laws in N for ``non smooth" potentials with strong short range correlations

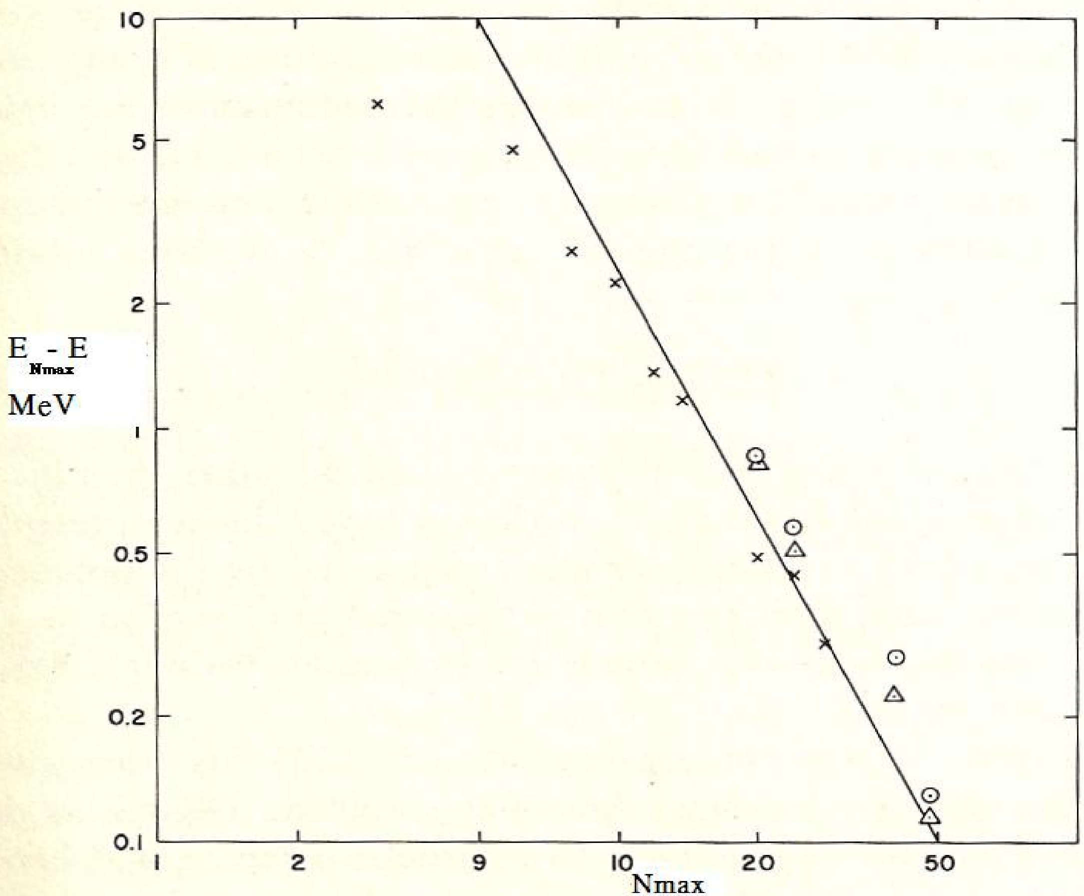
•exponential in N for ``smooth" potentials such as gaussians

HO basis: I. M. Delves, “Variational Techniques in the Nuclear Three-body Problem” in Advances in Nuclear Physics, Volume 5, ed. M. Baranger and E. Vogt (Plenum Press, New York, 1972) p.1-224, Simen Kvaal, “Harmonic oscillator eigenfunction expansions, quantum dots, and effective interactions”, Phys. Rev. B 80, 045321 (2009).

Hyperspherical Harmonics basis: T. R. Schneider, “Convergence of generalized spherical harmonic expansions in the three nucleon bound state”, Phys. Lett. 40B, 439 (1972).

However, the HO expansion basis has an intrinsic scale parameter $\hbar\omega$ which does not naturally fit into an extrapolation scheme based upon $\mathcal{N}$. Indeed the model spaces of these NCSM approaches are characterized by the ordered pair $(\mathcal{N}, \hbar\omega)$ where the basis truncation parameter $\mathcal{N}$ and the HO energy parameter $\hbar\omega$ are variational parameters. It is the purpose of this talk to summarize the properties of another ordered pair which more physically describes the nature of the model spaces and provides extrapolation tools which use $\mathcal{N}$ and $\hbar\omega$ on an equal footing

L.M Delves; in Advances
In Nuclear Physics vol 5 1972



$$E_{N_{max}} = E + P(N_{max})^{-2}$$

“nonsmooth potentials” like Yukawa

Fig. 8. **Convergence rates for variational calculations with a harmonic oscillator basis.** ⊙ Deuteron, Yamaguchi potential; × triton, Yamaguchi potential; and △ deuteron, Reid potential. Results taken from (JLS 70). The solid line has a slope of −2.0.

N_{max}

(JLS 70) is Jackson, Lande, & Sauer

“These results are independent of the dimensionality of the problem, that is, of the number of particles, provided that the appropriate N_{max} is used. … The extrapolated results of these authors have been used for E. On the logarithmic scale used, these differences are predicted by our crude theory to lie on a straight line of slope 2 for the Reid potential; it is not clear to what extent we should expect the nonlocal [separable] Yamaguchi potential to be `smooth'.”

Variational energy as a function of oscillator energy ħω for fixed number of quanta
Number of quanta increases by two for each curve

1969 H atom up to 10 quanta

M. MOSHINSKY

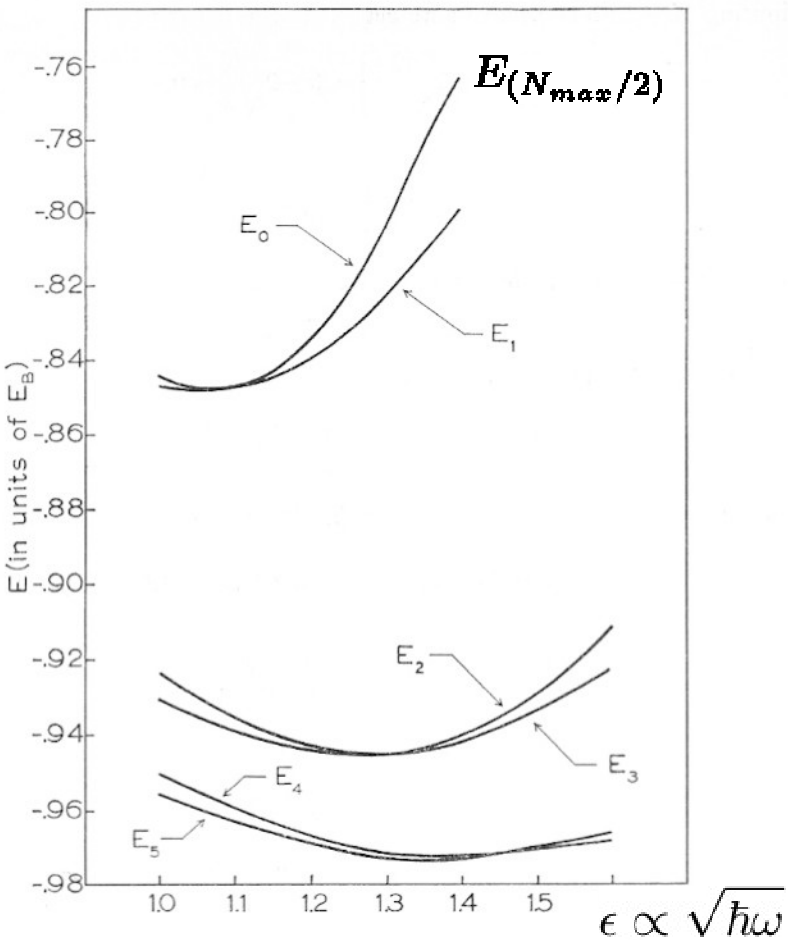
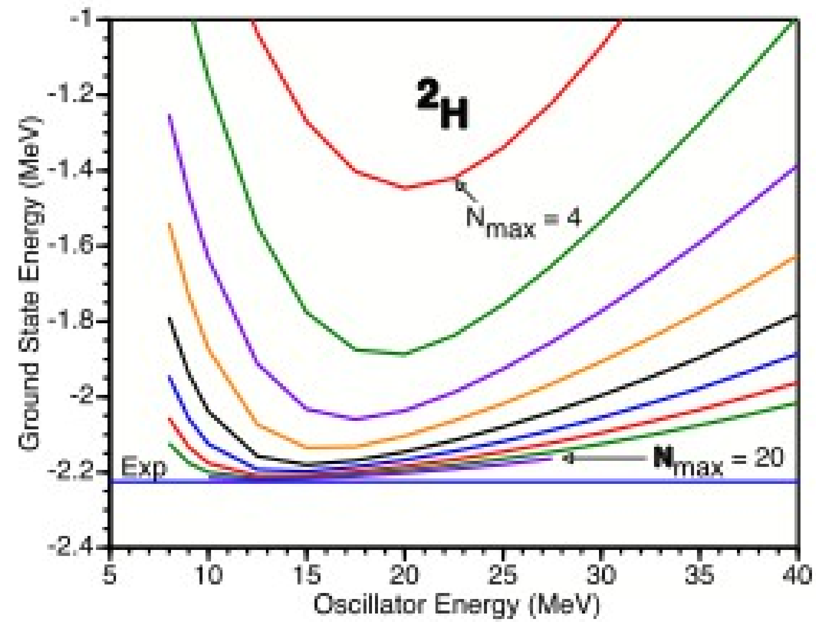


FIG. 1. Energy of the ground state of the H atom as a function of the parameter ε for the variational analysis discussed in Section 3. This energy $E_p(\varepsilon)$, $p = 0, 1, 2, 3, 4, 5$ is associated with a trial wave function $\psi_p = \Sigma_{n=0}^{p} a_n^{(p)} \mid n00\rangle$, where $\mid n00\rangle$ is a harmonic-oscillator state of frequency $\hbar\omega = (me^4/2\hbar^2)\varepsilon^2$.

2009 deuteron up to 20 quanta



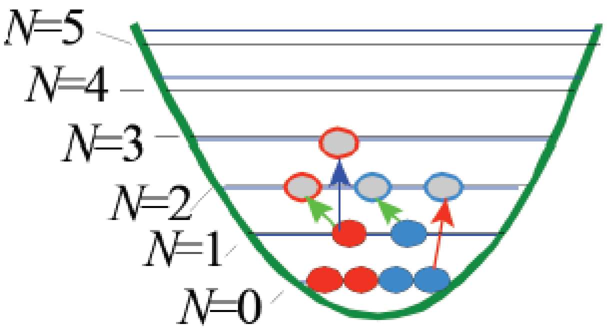
No-core full configuration method of
Maris,Vary,Shirokov

The No-Core Shell Model (NCSM)

Starting Hamiltonian is translationally invariant.

$$H_A = \frac{1}{A}\sum_{i<j}^{A}\frac{(\vec{p_i}-\vec{p_j})^2}{2m} + \sum_{i<j}^{A} V_{\mathrm{NN},ij}$$

NCSM has two parameters: Nmax and Ω



Provided interaction is "soft" we don't need to do any renormalization of interaction,

It's that "simple".

If we now use a single-particle basis, we have to remove the spurious CM states. Translational invariance is automatic if HO basis depends on Jacobi coordinates.

Advantage in m-scheme: Antisymmetry is easy to implement.

Disadvantage in m-scheme: Number of basis states is much larger than JT basis

Slide from Michael Kruse

Extrapolating with N_{Max}

Challenge: achieve numerical convergence for no-core Full Configuation calculations using finite model space calculations

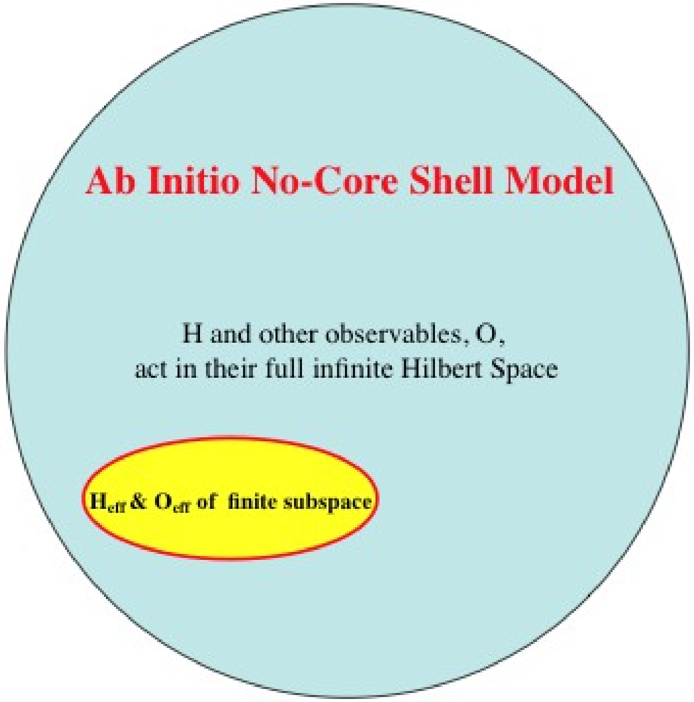
- Perform a series of calculations with increasing N_{max} truncation (while keeping everything else fixed)
- Extrapolate to infinite model space $\longrightarrow$ exact results
  - binding energy: exponential in N_{max}

$$E^{N}_{\text{binding}} = E^{\infty}_{\text{binding}} + a_1 \exp(-a_2 N_{\text{max}})$$

  - use 3 or 4 consecutive N_{max} values to determine $E^{\infty}_{\text{binding}}$
  - use $\hbar\omega$ and N_{max} dependence to estimate numerical error bars

Maris, Shirokov, Vary, Phys. Rev. C79, 014308 (2009)

Slide by Pieter Maris



This truncation/extrapolation scheme is essentially that of the earlier few-body variational studies

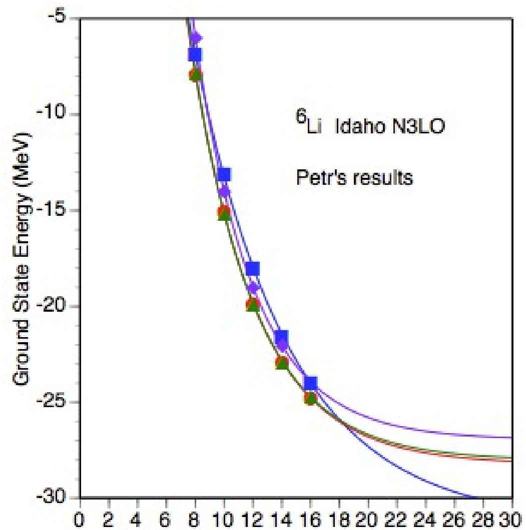
Assumes that the boundary of finite subspace is defined only by N_{max}

implication: $\hbar\omega$ is an inessential complication

Not the case! The use of HO single particle orbitals means that the system is limited to a region in coordinate space whose size is governed by the parameter of the HO basis: $\hbar\omega$

The finite model space is characterized by two parameters:
N_{max} and $\hbar\omega$

Troubles with N_{max} extrapolation



hw = 25 MeV E(N=infity) is -31.26 MeV

hw = 33 MeV E(N=infity) is -28.22 MeV

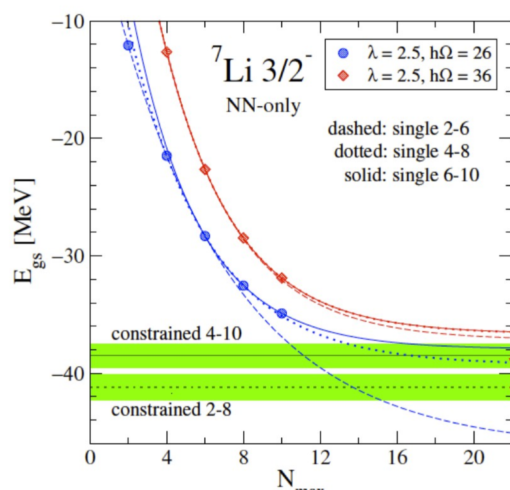
hw = 34 MeV E(N=infity) is -28.02 MeV

hw = 40 MeV E(N=infity) is -26.94 MeV

Petr's extrapolation is -28.2 MeV
Fig. 6 of Navratil & Caurier
PHYSICAL REVIEW C **69**, 014311 (2004)

Nmax	Lambda(33)	Lambda(25)
4	449 MeV/c	391 MeV/c
6	513	
8	570	
10	622	
12	670	
14	715	
16	757	659

Chiral NN interaction, λ = 2.5 fm-1
Extrapolation A'



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Effective Field Theory (EFT)

In a field theory one ***never*** has access to the "full" Hilbert space. Experiments only probe a region of momenta. Nature is quantum mechanical. So to develop a theory for such a region we must pose a model space. For smallest errors the model space should be as big, if possible, as the region one is interested in.

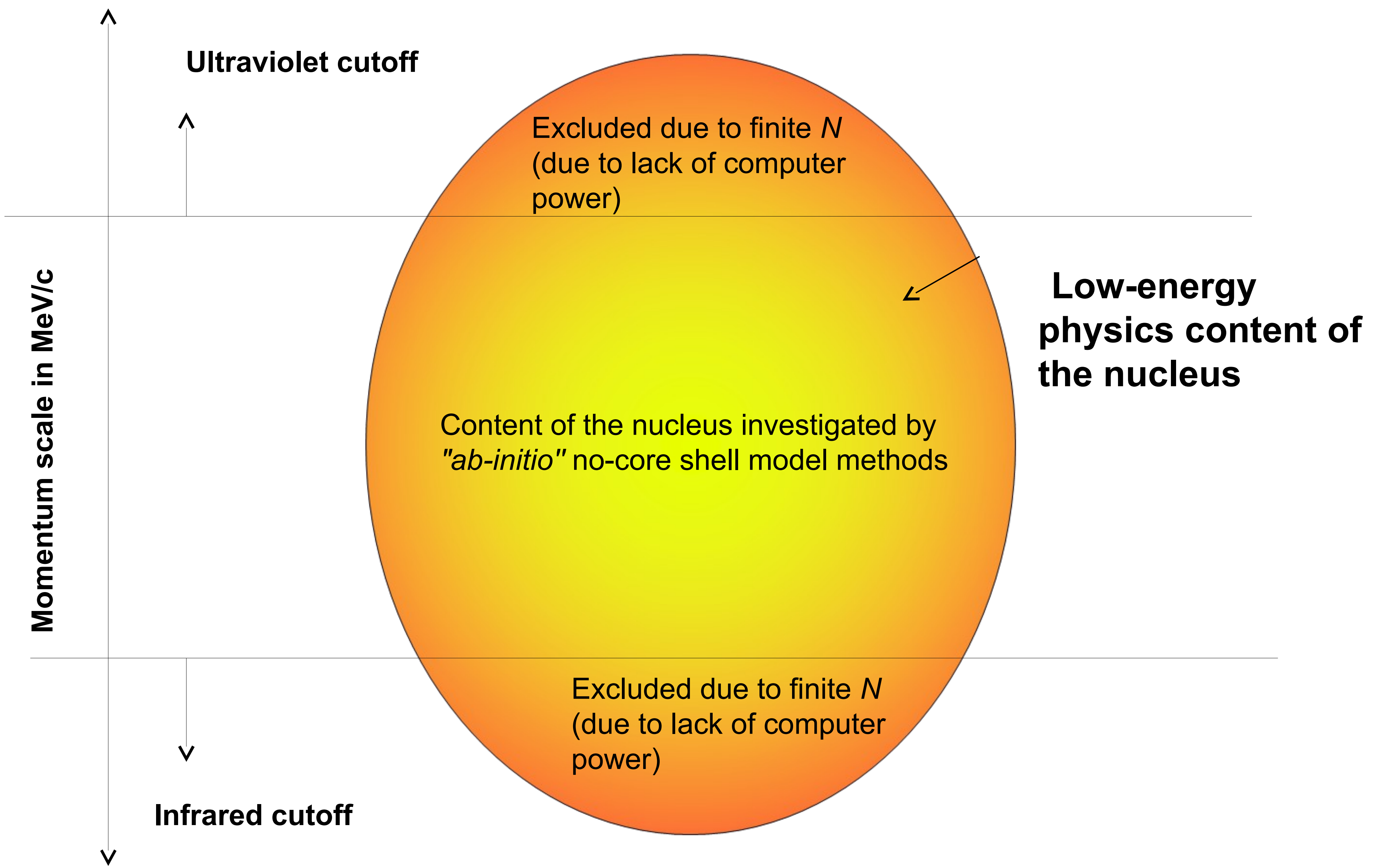
The parameter of the projection operator P into the model space must have a dimension. Call the parameter Λ, the ultraviolet cutoff and take it to be a momentum.

Model space can be arbitrary but observables calculated within it cannot.
The Hamiltonian operator of the model space must depend on Λ in such a way that observables at momenta $Q \ll \Lambda$ are independent of how P is chosen, and in particular, independent of Λ.

Arizona program: formulate a nuclear EFT in an HO basis as an efficient way of reaching larger nuclei. To limit the number of one-particle states introduce a parameter λ, an infrared cutoff in addition to Λ, so that observables at momenta $Q \gg \lambda$ are independent of λ. That is, the values of Λ and λ control the size of the model space and the projection operators $P(\Lambda)$ and $P(\lambda)$ define the boundaries of the model space.

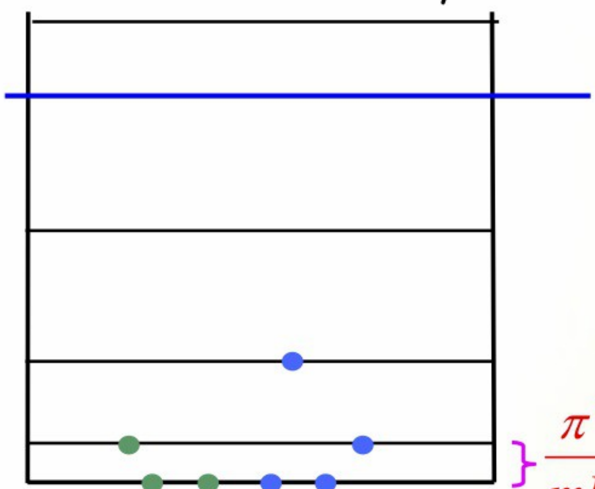
van Kolck, Barrett, Stetcu, Rotureau, Yang

My more modest goal: can EFT motivate and shape an extrapolation to the infinite basis limit for the HO basis calculations called NCSM or NCFC which utilize "realistic" nuclear interactions fit to data, not in a clearly defined model space, but in free space?


Ultraviolet cutoff
Excluded due to finite *N* (due to lack of computer power)
Low-energy physics content of the nucleus
Momentum scale in MeV/c
Content of the nucleus investigated by *"ab-initio"* no-core shell model methods
Excluded due to finite *N* (due to lack of computer power)
Infrared cutoff

Popular examples

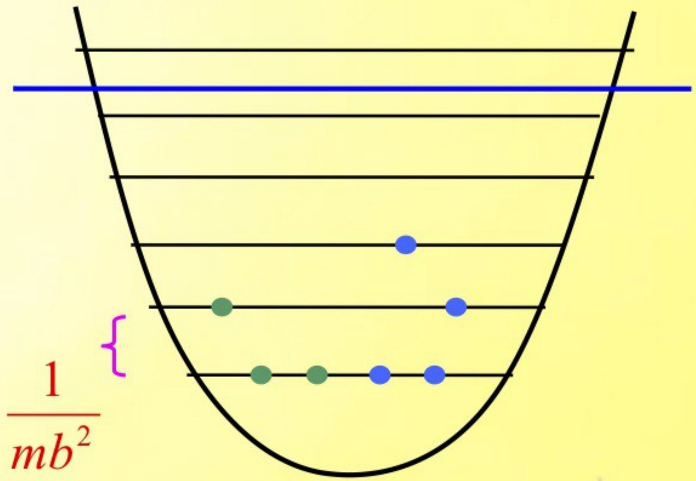
Lattice Box		Harmonic-Oscillator Box
"Lattice Field Theory"		"No-Core Shell Model"



energy

$$\frac{N^2\pi^2}{mL^2} \qquad \frac{\Lambda^2}{2m} \qquad \frac{N_{max}}{mb^2}$$

$$\frac{\pi^2}{mL^2} \qquad \frac{\lambda^2}{2m} \qquad \frac{1}{mb^2}$$



$$L = Na$$

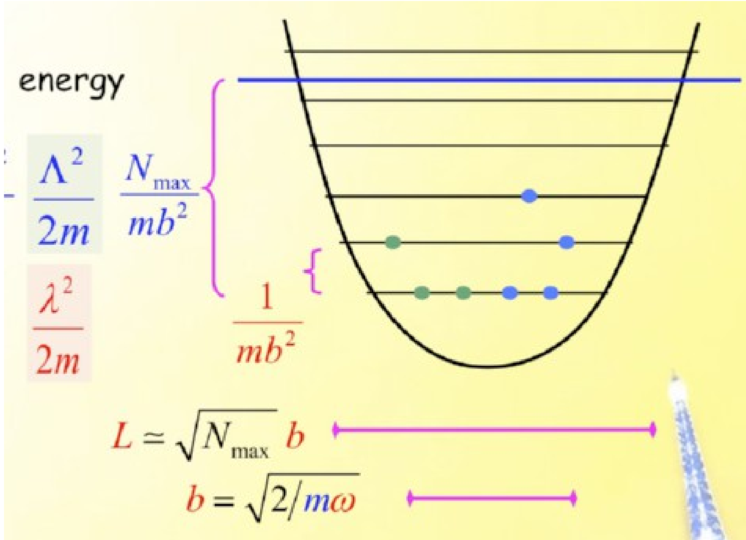
$$L \simeq \sqrt{N_{max}}\, b$$

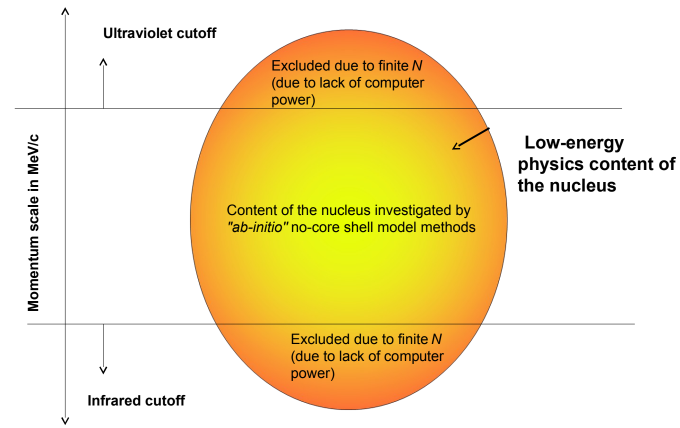
$$b = \sqrt{2/m\omega}$$

nuclear matter	Müller *et al.* '99
few nucleons	Lee *et al.* '05 ...
atomic matter	Bulgac *et al.* '06 ...
few atoms	Kaplan *et al.* '10 ...

finite nuclei	Stetcu *et al.* '06 ...
few atoms	Stetcu *et al.* '07 ...

Slide by Bira van Kolck





Define a UV momentum cutoff Λ analogous to continuum Λ in which the particles are not confined:

$$\Lambda = \sqrt{m_N (N_{Max} + 3/2)\hbar\omega}$$

A high momentum cutoff corresponding to the maximal non-infinite momentum in the bound system.

Interpret behavior of variational energy of system as more basis states are added as the running of an observable with the variation (increase) of the UV cutoff of model space

Confinement to a volume because ħω>0 means the energy levels are quantized. The associated momenta cannot take on continuous values so that the model space has an infrared (IR) momentum cutoff λ.

Define $$\lambda = \sqrt{(m_N \hbar\omega)}$$

A low momentum cutoff corresponding to the minimal non-zero momentum in the bound system.

Another discretization scheme: QCD on a 4-dimensional lattice

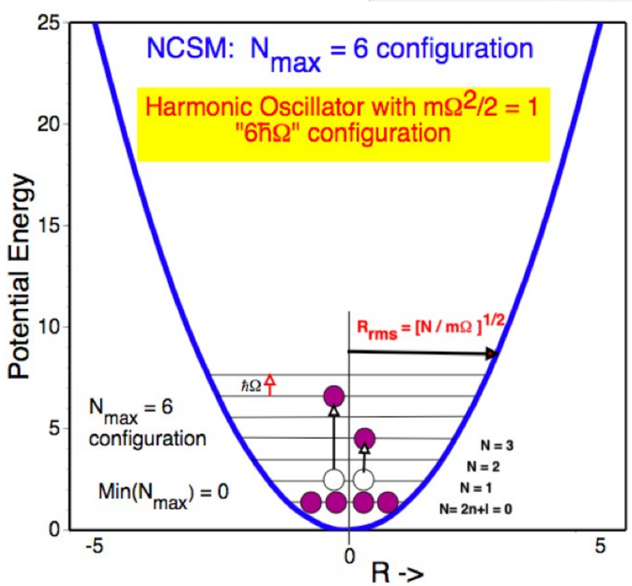
Continuum QCD simulated on a lattice has a model space with two cutoffs
UV cutoff   Λ~1/a where a is lattice spacing
IR cutoff   λ~1/L where L is the size of the lattice
a must be small enough to simulate the continuum
L must be large enough to contain the system

Suggests another possible IR cutoff for a HO basis

$$\lambda_{SC} = \sqrt{(m_N \hbar\omega)/(N_{Max} + 3/2)}$$

This IR cutoff is the inverse of the rms radius of the highest single particle state in the basis, i.e. the maximal radial extent needed to encompass the system

$$\lambda_{SC} = 1/(\sqrt{N_{Max} + 3/2}b)$$ where $$b = \sqrt{\frac{\hbar}{m_N \omega}}$$

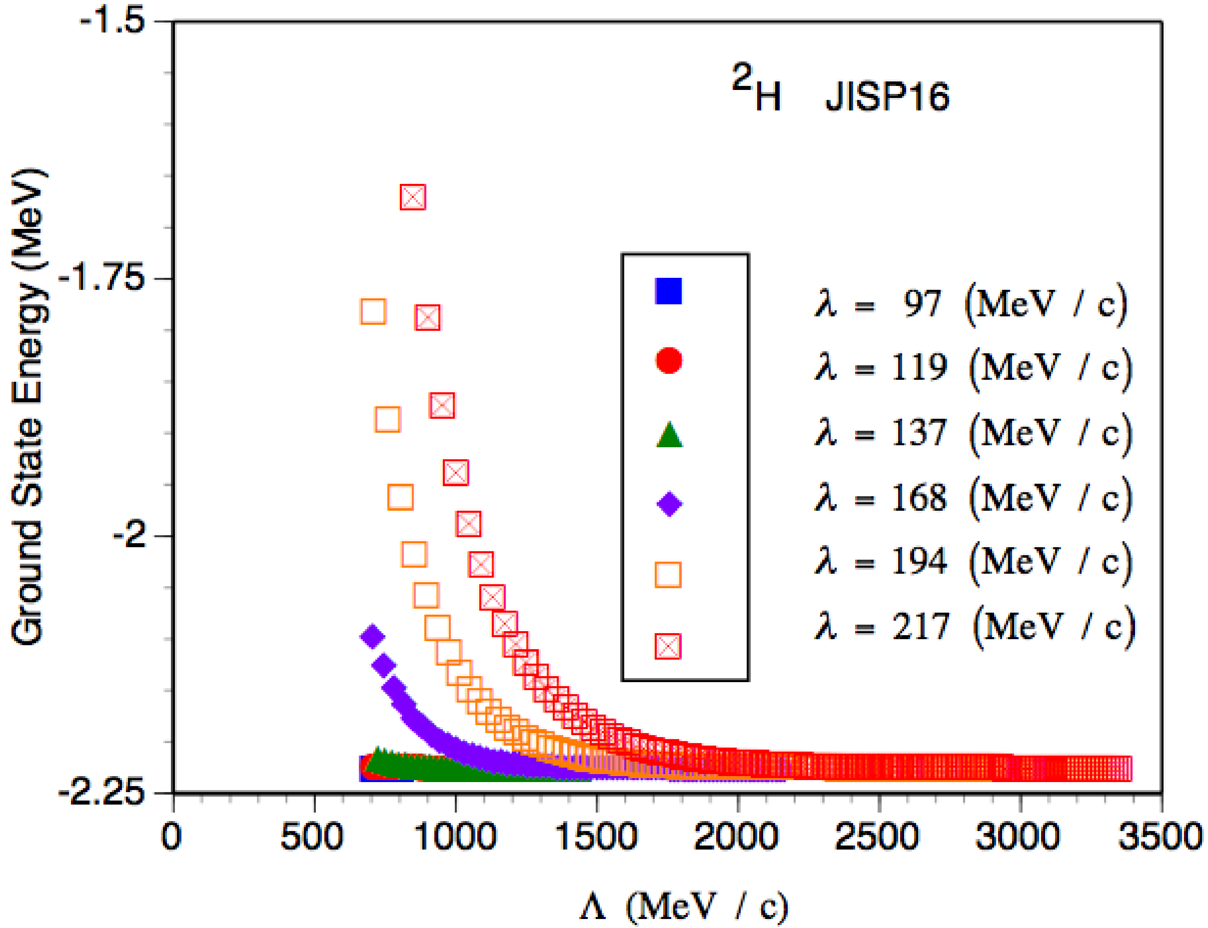


Note 1 $$\lambda_{SC} = \lambda^2/\Lambda$$

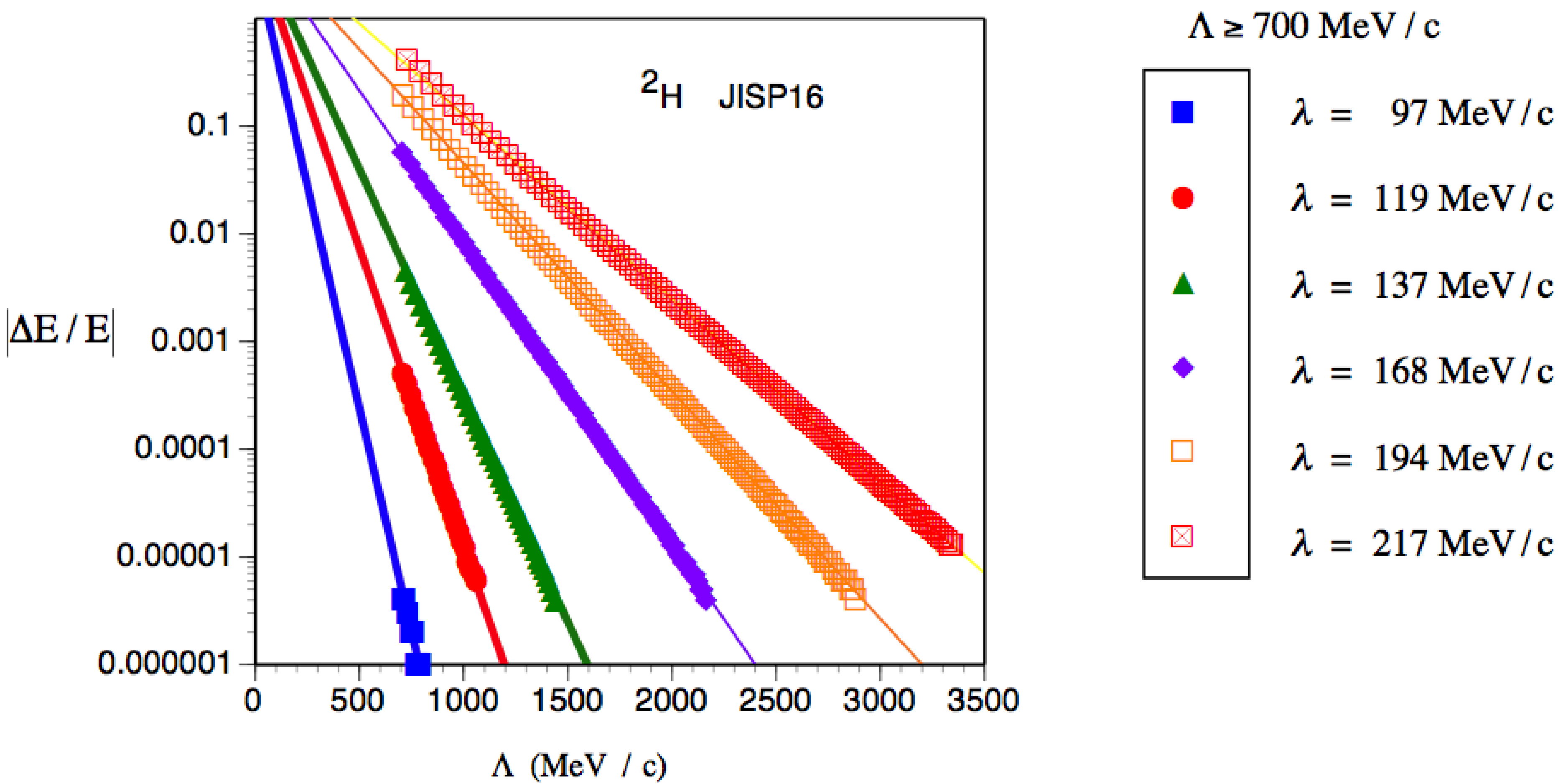
Note 2
$$N_{max} + 3/2 = \Lambda^2/\lambda^2 = \Lambda/\lambda_{sc}$$

A ``P(N_{max}) " mixes up two dimensionful cutoffs

Test model space cutoffs with deuteron calculation done with defined N_{max} and ħω convergence is clear as N_{max} goes to 238



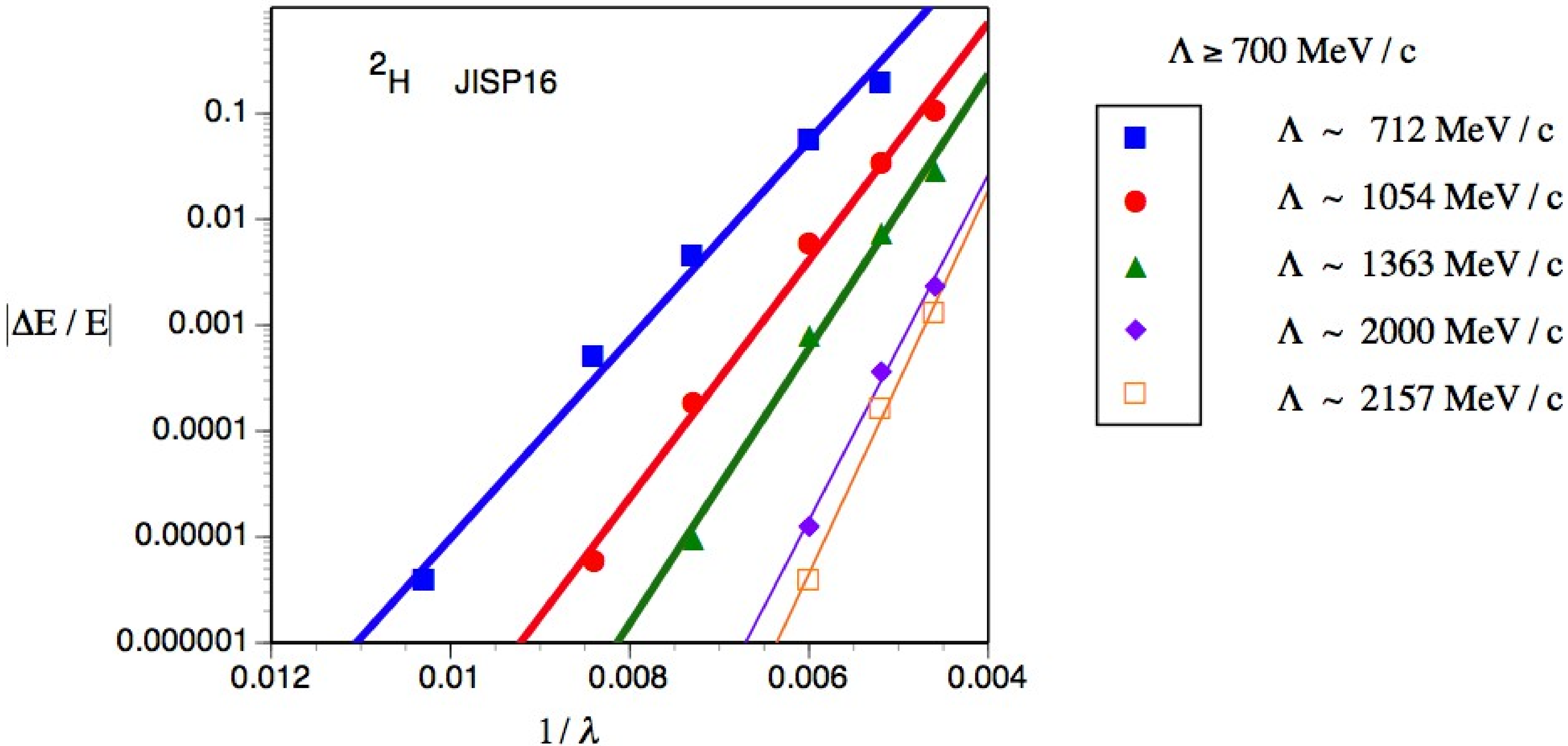
$\lambda_{IR} \equiv \lambda$ acts as an IR cutoff should!



As the ultraviolet cutoff increases, the fractional difference between calculated E(Λ, λ) and an accepted-as-converged E, decreases.

Alternatively, the plot can be read the other way, where if we fix the UV Λ, the results improve as we lower the IR cutoff λ.

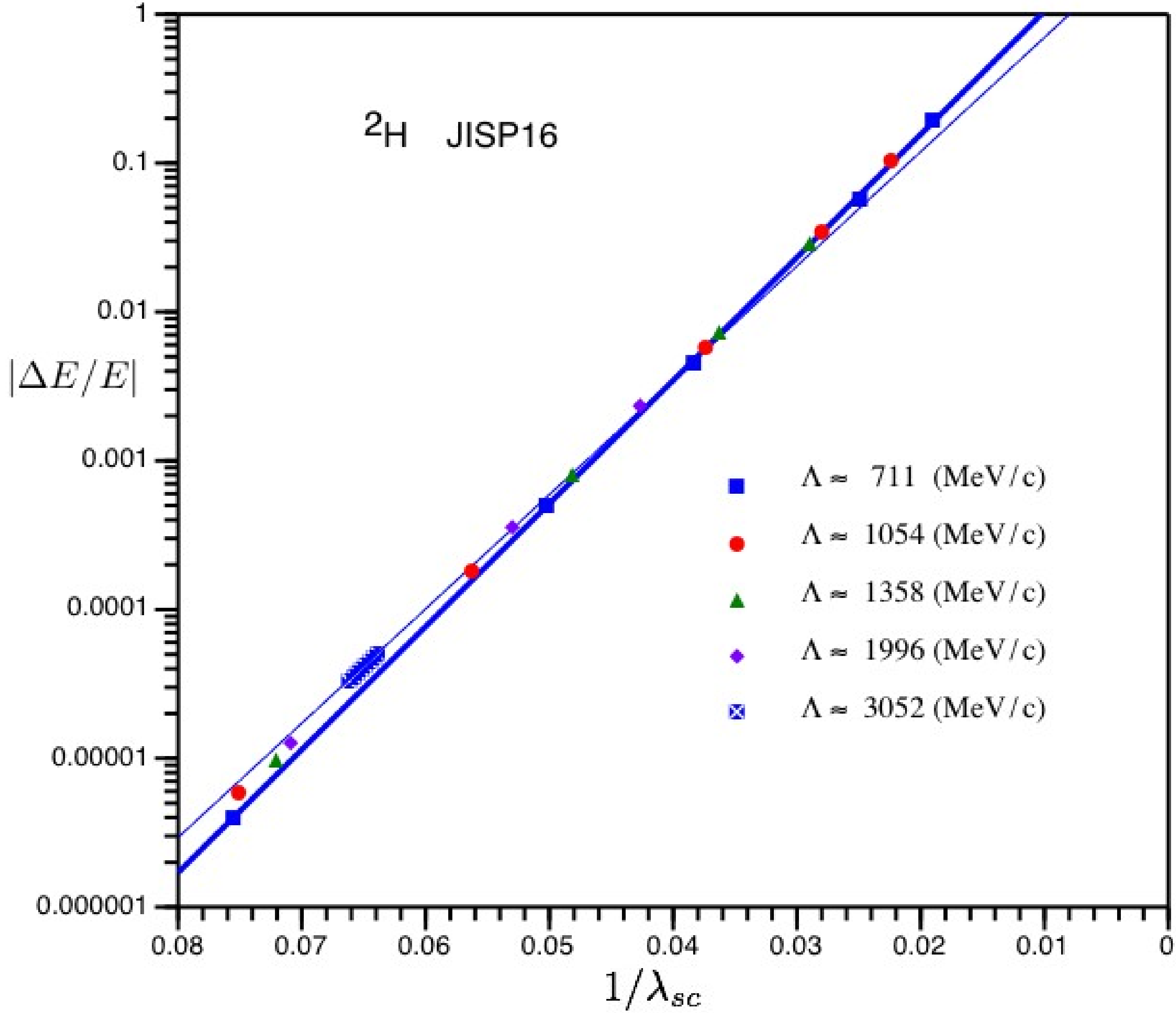
Λ acts as an UV cutoff should!



small λ

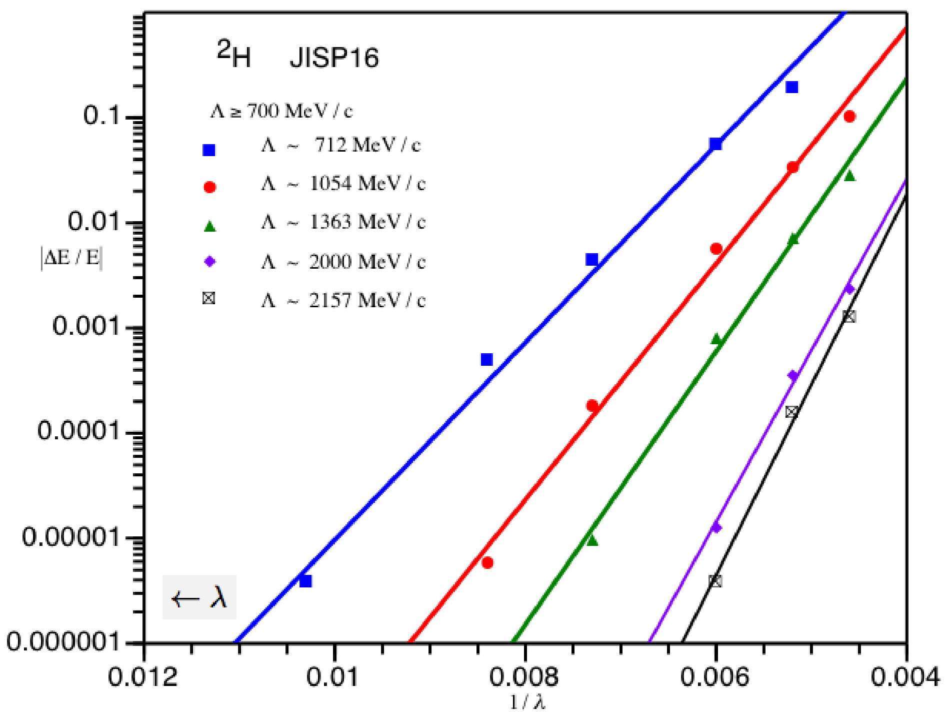
large λ

Result scales with $1/\lambda_{sc} = \Lambda/\lambda^2$ almost a universal behavior



$1/\lambda_{sc}$ has units of a length. $1/\lambda_{sc}$ is the maximal radial extent needed to encompass the system. One could call this radius ``L" if one wanted a different name for $1/\lambda_{sc}$.

Remove IR effects by decreasing value of IR momentum cutoff in the function chosen as an extrapolator whilst keeping the UV cutoff undisturbed.



- Extrapolator is clearly the exponential function.
- $\frac{E(\lambda)-E(\lambda=0)}{E(\lambda=0)} \equiv \frac{\Delta E}{E} = A\exp(-B/\lambda)$
- B is a function of the UV cutoff Λ
- The IR cutoff *cannot* be aware of the UV cutoff.
- Remove dependence upon Λ
- $\lambda = \sqrt{\Lambda\lambda_{sc}} \Longrightarrow$
- $\exp(-B/\lambda) = \exp(-B/\sqrt{\Lambda\lambda_{sc}}) = \exp\left(-\frac{B/\sqrt{\Lambda}}{\sqrt{\lambda_{sc}}}\right)$

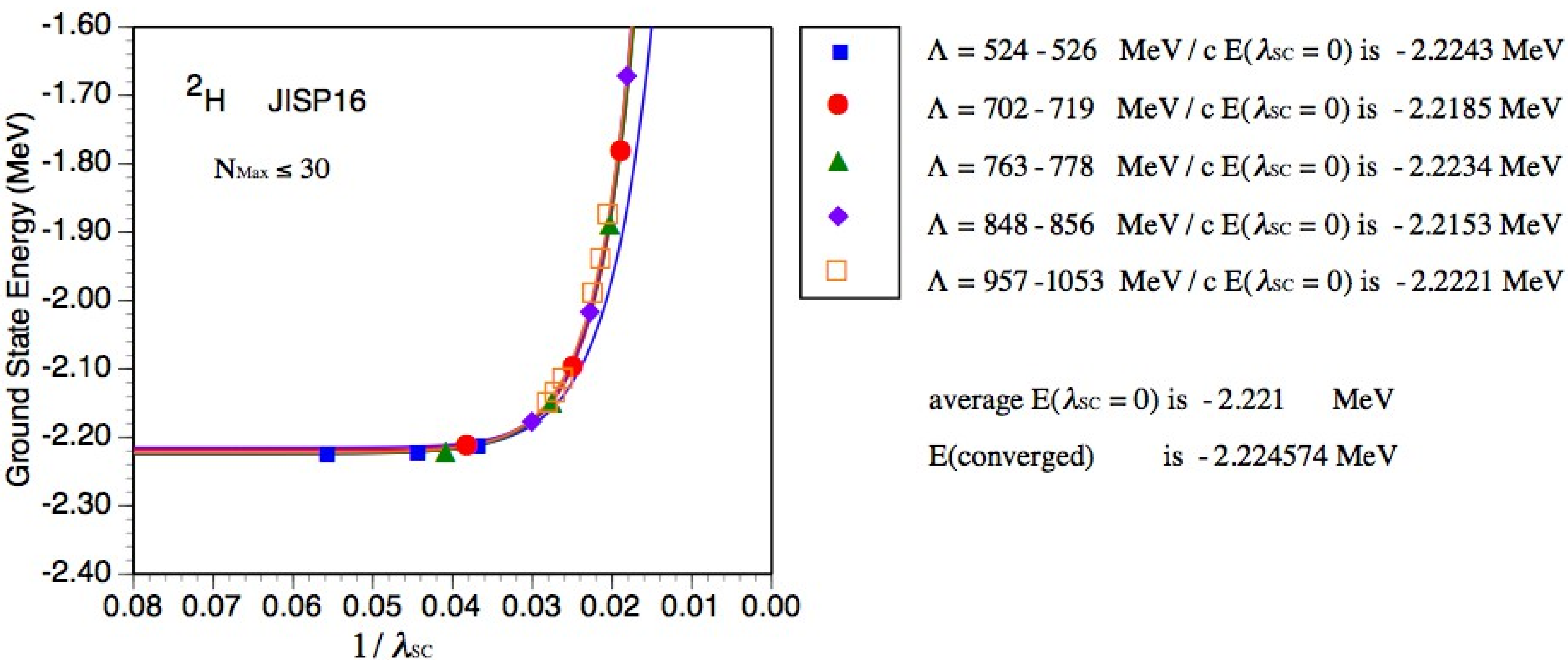
$B/\sqrt{\Lambda}$ (i.e., multiplier of $1/\sqrt{\lambda_{sc}}$) is constant to within 5 %.

The momentum cutoff λ will remove IR effects. Indeed, any momentum cutoff $\lambda_{sc} \leq \lambda_{IR} \leq \Lambda$ will remove IR effects, but the IR regulator which is independent of the UV cutoff is some function of λ_{sc}. It is λ_{sc} which causes the IR effects and one does not need to decrease a IR cutoff below that of λ_{sc} to remove IR effects (i.e. extrapolate to zero).

Success! UV and IR cutoffs identified as $N_{max} \to 238$

Are cutoffs of any use for approachable N_{max}?

$$E(\lambda_{sc}) = A\exp(-B/\lambda_{sc}) + E(\lambda_{sc}=0)$$



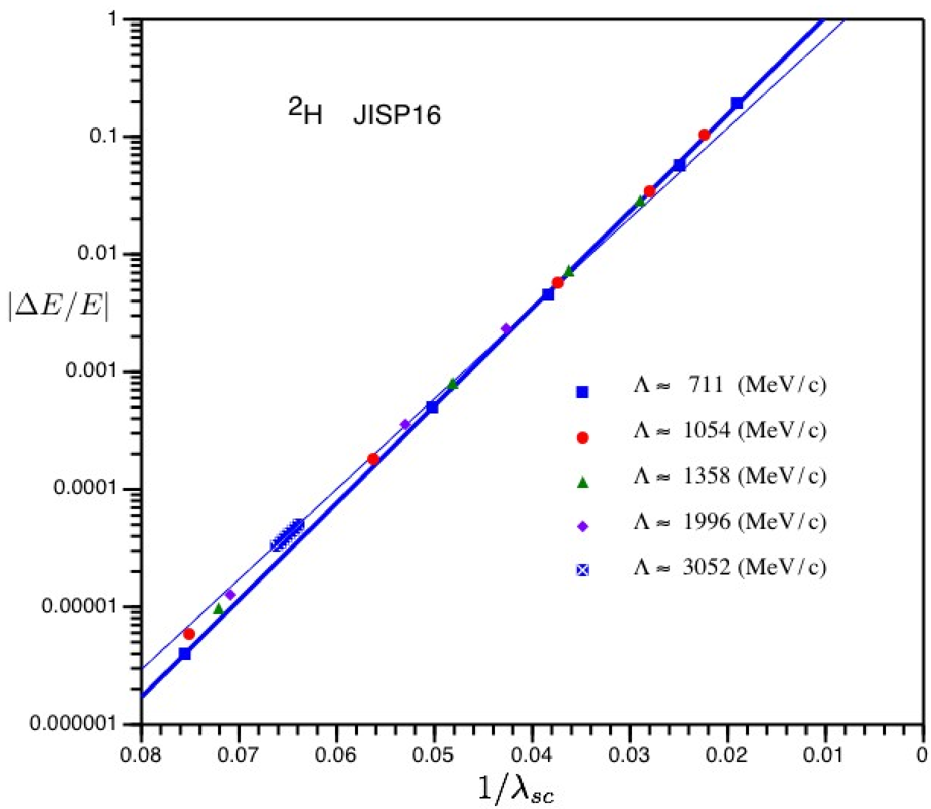
average $E(\lambda_{sc} = 0)$ is -2.221 MeV

E(converged) is -2.224574 MeV

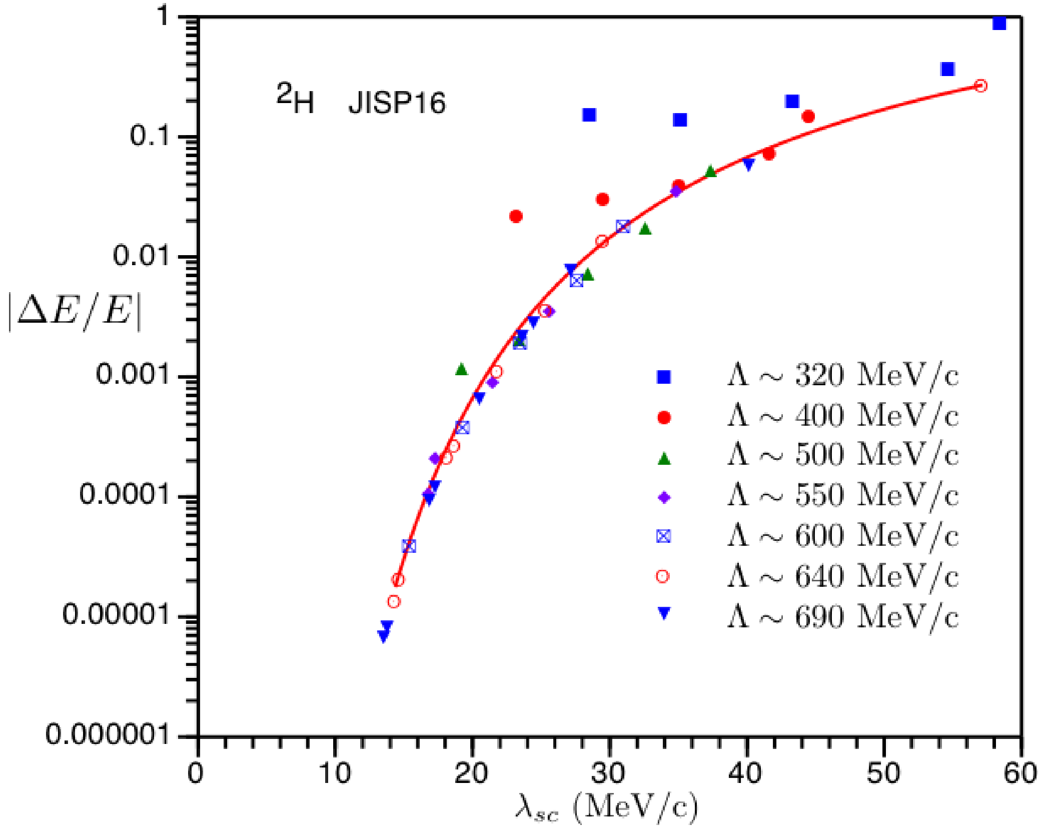
Note: This is not the usual extrapolation in N_{max} (with some prescription for ħω) because

$$\begin{aligned} 1/\lambda_{sc} &= \Lambda/\lambda^2 \\ &= \sqrt{(N_{max}+3/2)/(m_N\hbar\omega)} \\ &\propto \sqrt{N_{max}/(m_N\hbar\omega)} \end{aligned}$$

$\boxed{N_{max} \text{ and } \hbar\omega \text{ on an equal footing}}$

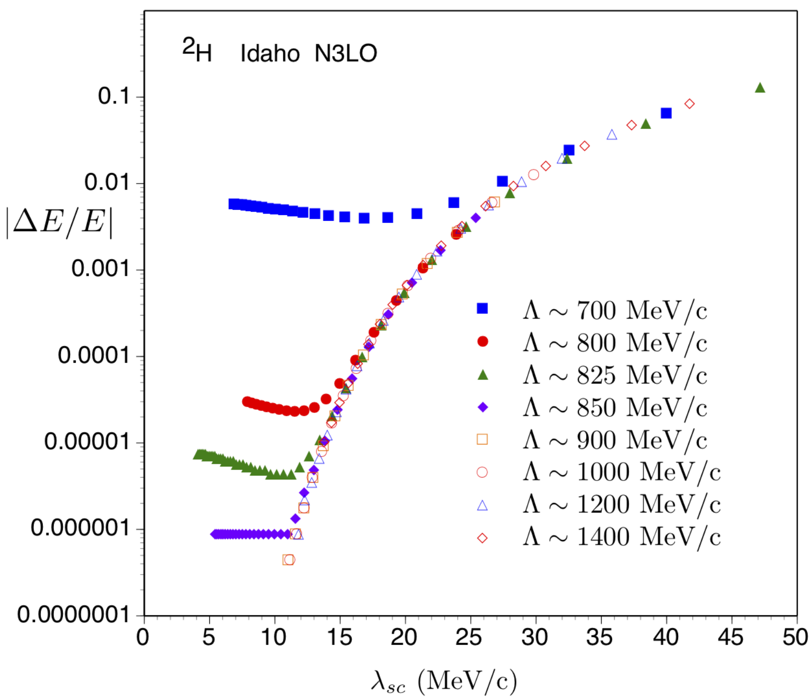


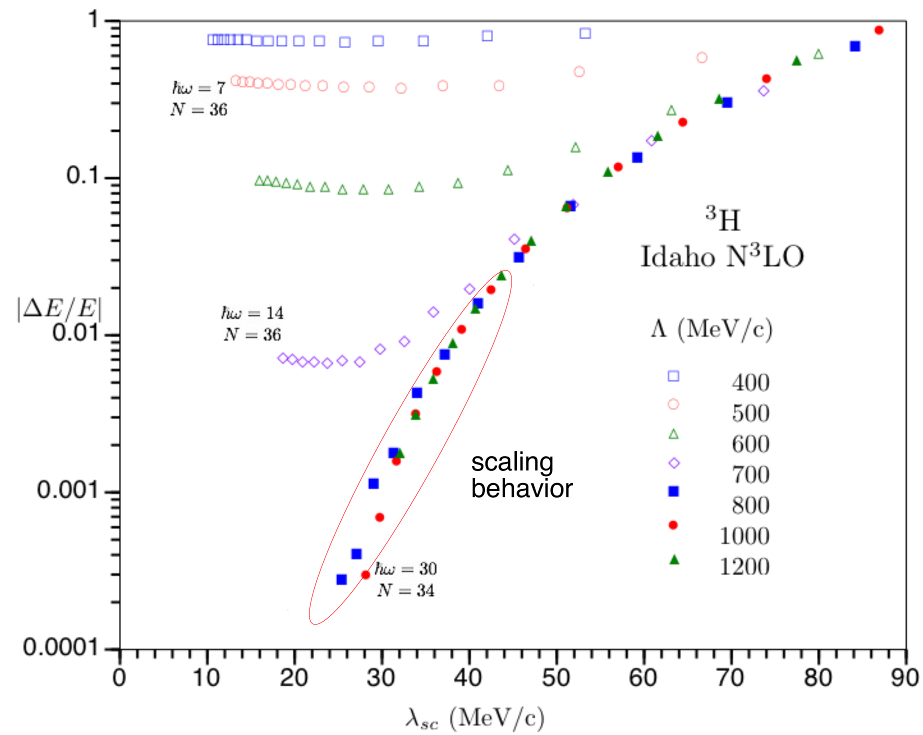
Λ > 700 MeV/c



Λ < 700 MeV/c

Running of $|\Delta E/E|$ with IR cutoff suggests an intrinsic UV scale of the NN interaction





$|\Delta E/E|$ does not go to zero unless $\Lambda > \Lambda^{NN}$ where Λ^{NN} is some UV regulator scale of the NN interaction

For $\Lambda < \Lambda^{NN}$ there are missing contributions of size $|\Lambda - \Lambda^{NN}|/\Lambda^{NN}$ so "plateaus" appear as IR cutoff approaches 0.

Rise of plateaus suggests corrections are needed to Λ and λ_{sc}, which are def ned only to leading order in λ_{sc}/Λ.

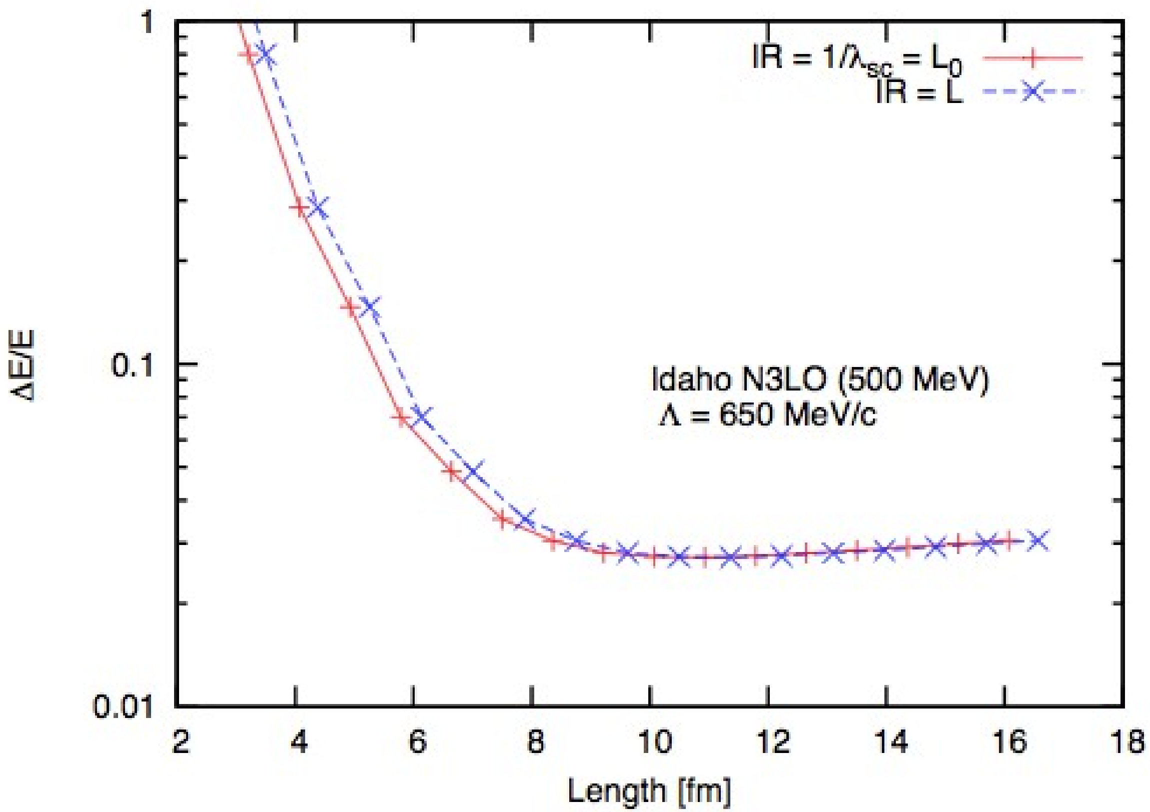
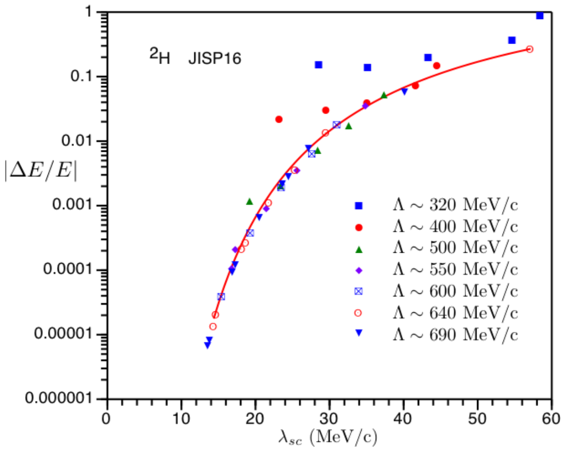
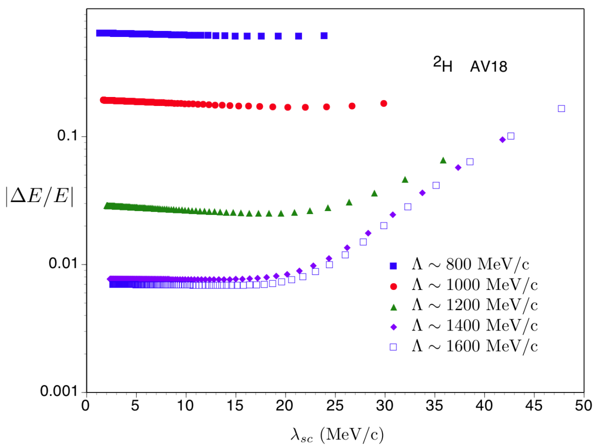


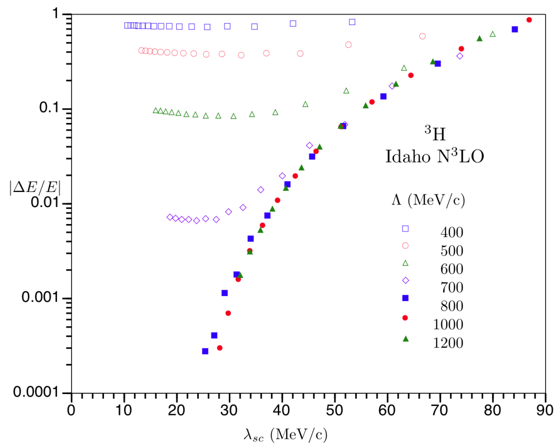
Figure 1: The plot shows the famous plateaus for $\Lambda = 650$ MeV/c. Note that there is no difference between using the original IR definition L_0 or the 'corrected' IR L.

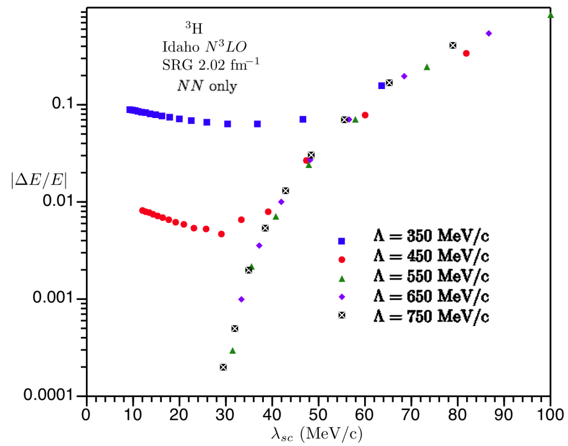
Michael Kruse-private communication
L_2 doesn't remove plateaus either

Intrinsic UV scale depends on the NN interaction









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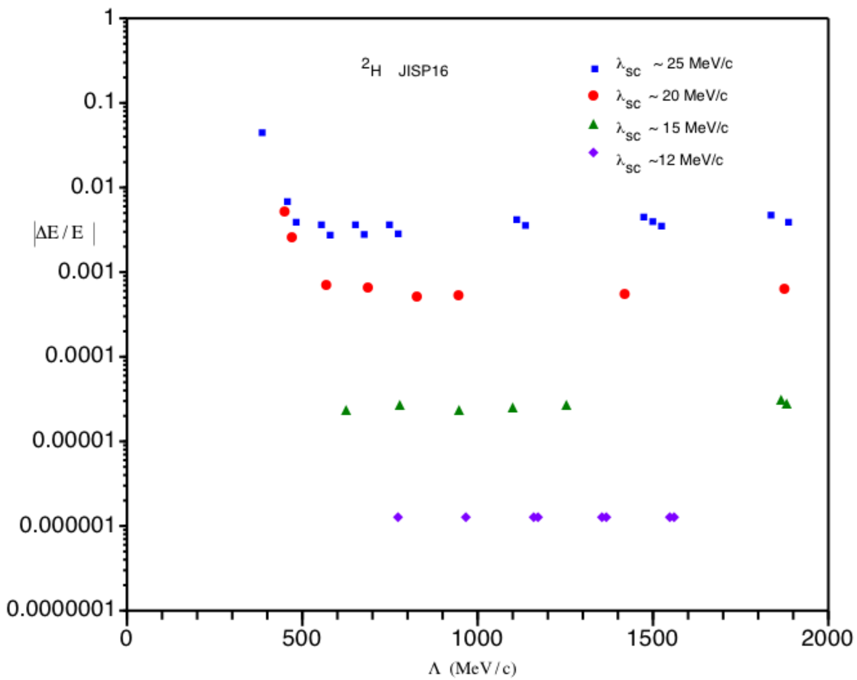
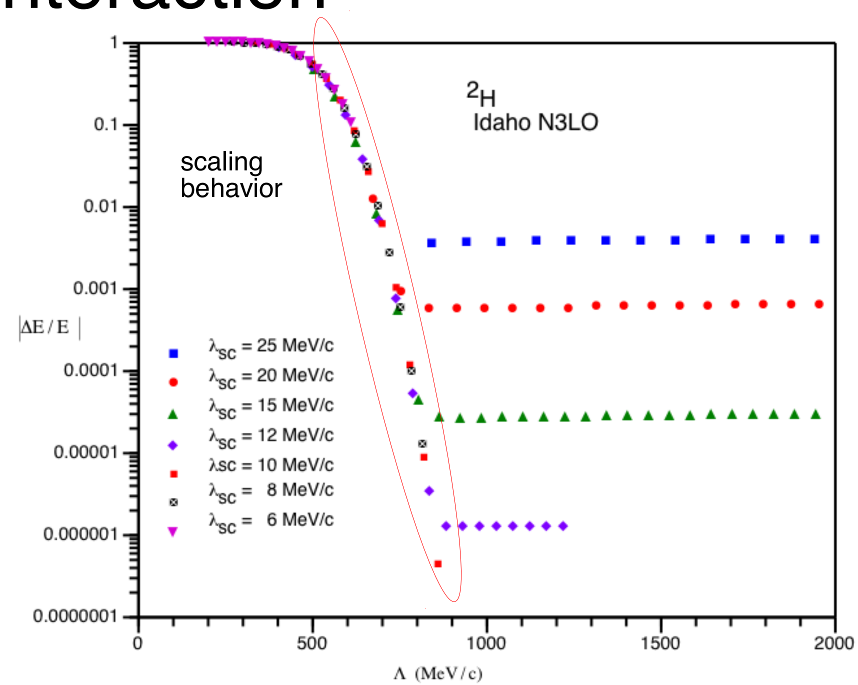
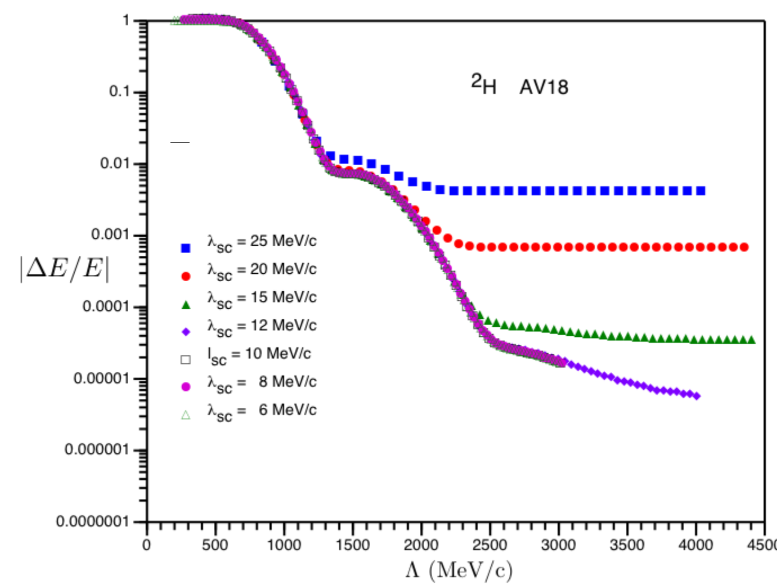




Plateau's ascribed to another "missing contributions" argument.



$|\Delta E/E|$ does not go to zero unless $\lambda_{sc} \leq \lambda_{sc}^{NN}$
where λ_{sc}^{NN} is some IR regulator scale of the NN interaction.

Value of λ_{sc}^{NN} is consistent with lowest energy conf guration described by NN interaction;
e.g. deuteron binding momentum Q = 45 MeV/c,
or average of inverse scattering lengths 16 MeV/c

Are deduced values of intrinsic Λ and λ_{sc} consistent with expectations?

S wave parts of JISP16 potential fit to data in a HO space of N=8 and $\hbar\omega$ = 40 MeV.

JISP16 regulator scales of $\lambda_{sc}^{NN} \sim 63$ MeV/c and $\Lambda^{NN} \sim 600$ MeV/c.

In practice, UV region seems already captured at Λ > 500-550 MeV/c.

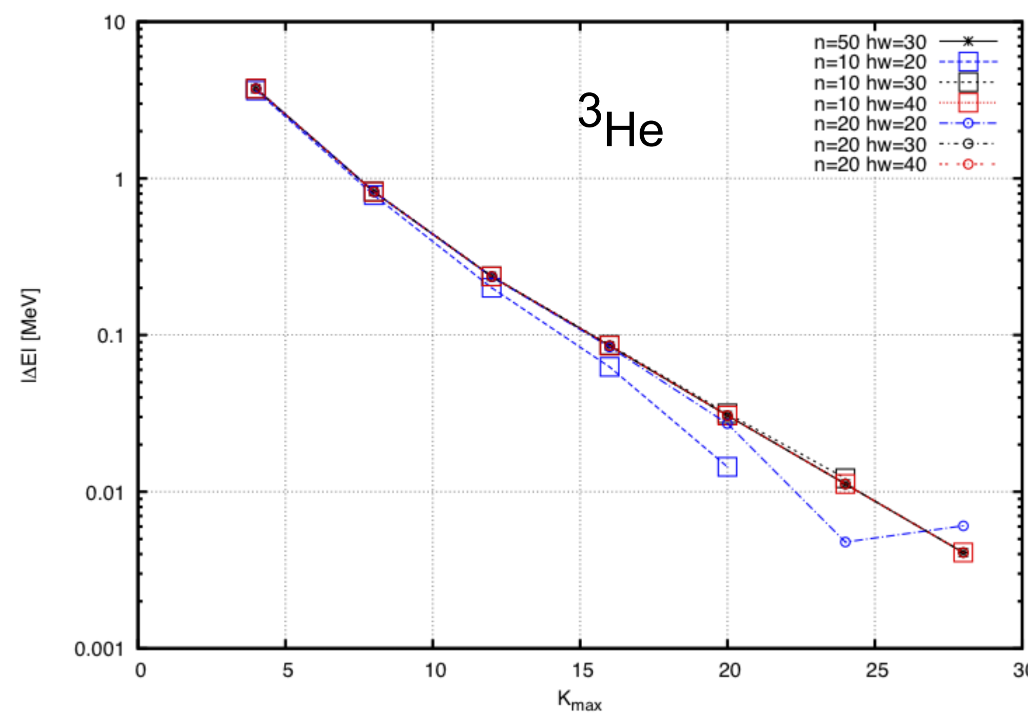
Idaho N^{3}LO potential fit to data in momentum space.
Gaussian regulator function with cutoff parameter of 500 MeV/c

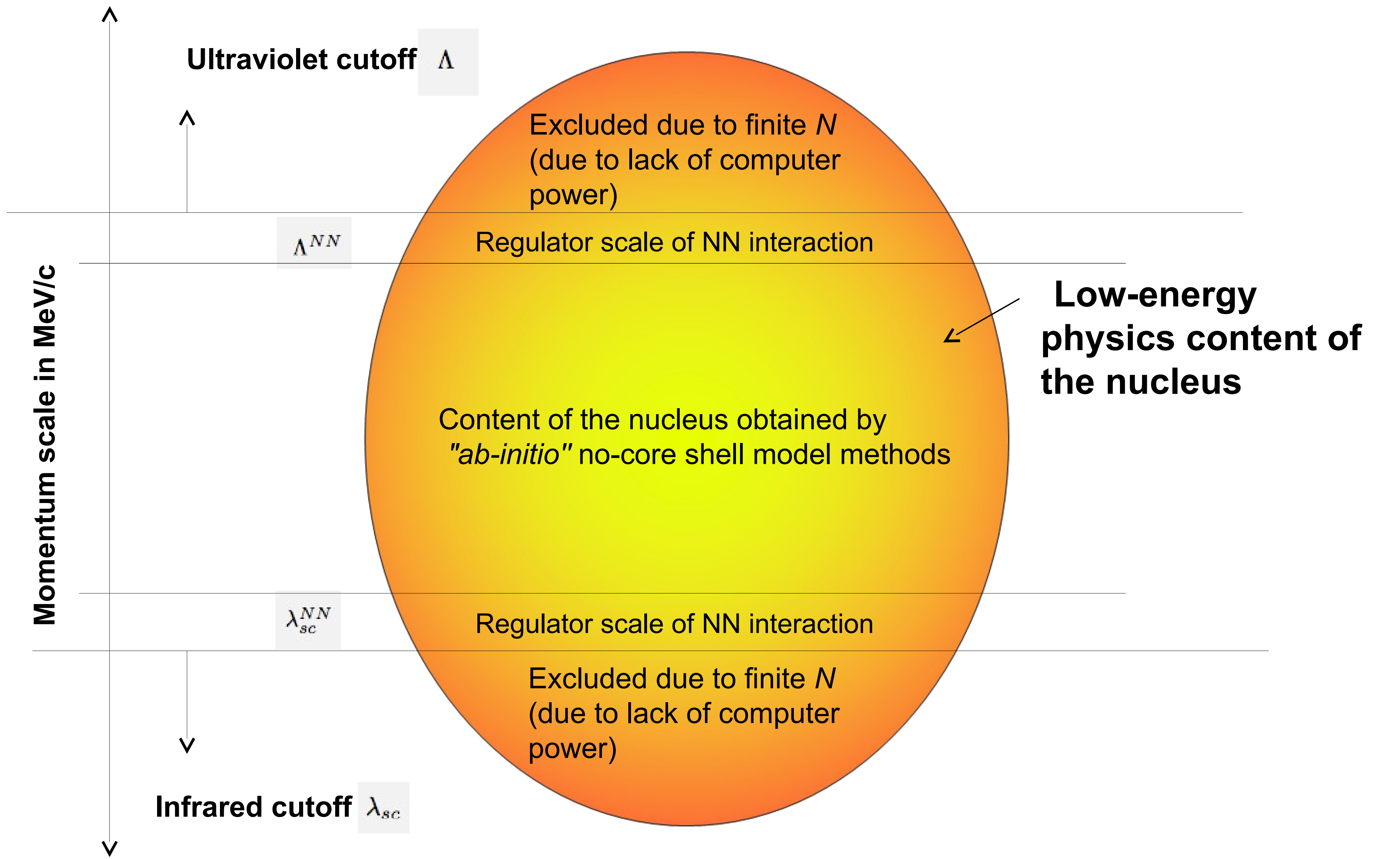
How to compare apples and oranges?

Represent this potential in a HO basis.
Barnea et al PRC 81, 064001 (2010)
Nir Barnea, private communication

Idaho N^{3}LO regulator scales of
$\lambda_{sc}^{NN} \sim$ 21-42 MeV/c
$\Lambda^{NN} \sim$ 900-1100 MeV/c.

In practice, UV region seems already captured at Λ > 800 MeV.




Ultraviolet cutoff Λ
Excluded due to finite *N* (due to lack of computer power)
Λ^{NN}
Regulator scale of NN interaction
Low-energy physics content of the nucleus
Momentum scale in MeV/c
Content of the nucleus obtained by *"ab-initio"* no-core shell model methods
λ_{sc}^{NN}
Regulator scale of NN interaction
Excluded due to finite *N* (due to lack of computer power)
Infrared cutoff λ_{sc}

How do intrinsic regulator scales determine needed values of N and hbar-omega for a converged result?

$$\begin{aligned} \Lambda/\lambda_{sc} &= N + 3/2 \\ (\Lambda\lambda_{sc})/m_N &= \hbar\omega \end{aligned}$$

$$\Lambda \geq \Lambda^{NN} = 800$$

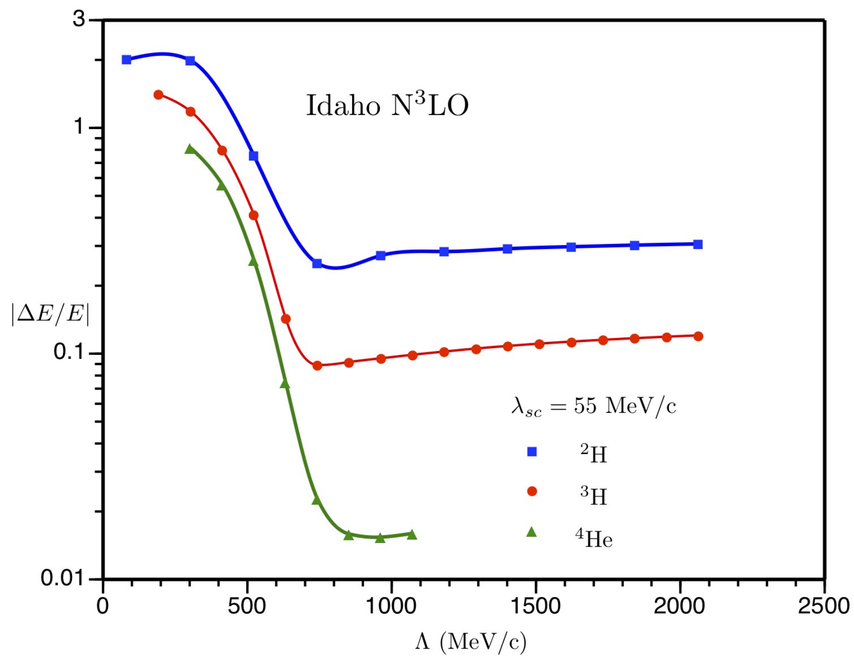
$\lambda_{sc} \approx 10$	$\lambda_{sc} \approx 20$	$\lambda_{sc} \approx 40$
$N \geq 80$	$N \geq 40$	$N \geq 20$
$\hbar\omega \succeq 8$	$\hbar\omega \succeq 16$	$\hbar\omega \succeq 32$

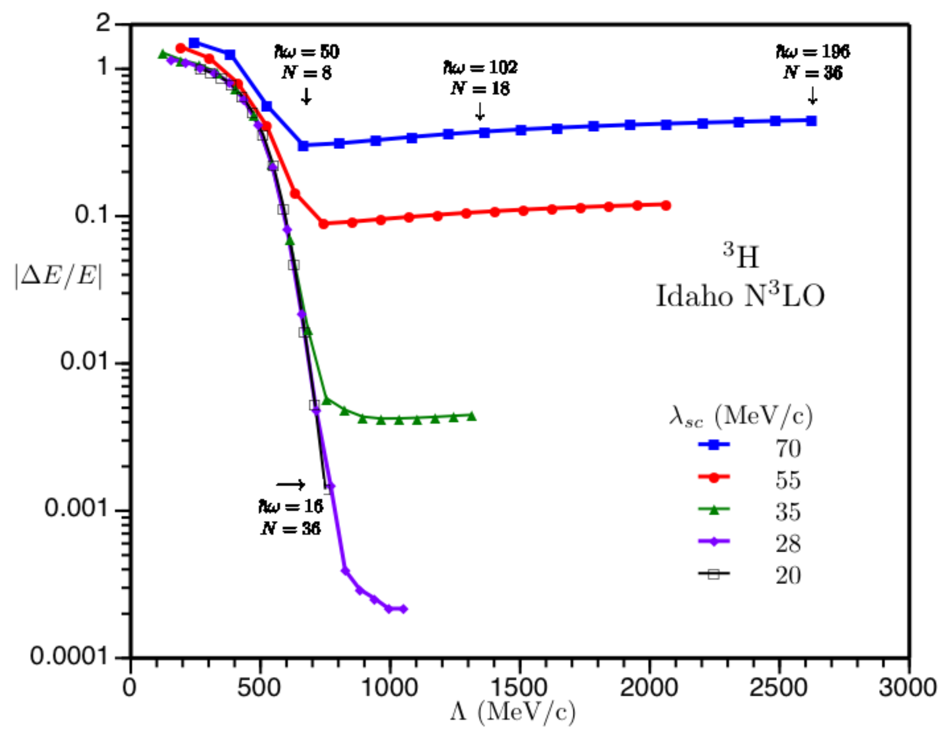
$$\Lambda \geq \Lambda^{NN} = 500$$

$\lambda_{sc} \approx 10$	$\lambda_{sc} \approx 20$	$\lambda_{sc} \approx 40$
$N \geq 50$	$N \geq 25$	$N \geq 12$
$\hbar\omega \succeq 5$	$\hbar\omega \succeq 10$	$\hbar\omega \succeq 20$

Conclusion: One must extrapolate for all but the lightest nuclei

For fixed λ_{SC} result does NOT improve with increasing Λ, if $\Lambda \geq \Lambda^{NN}$ ~ 800 MeV/c !

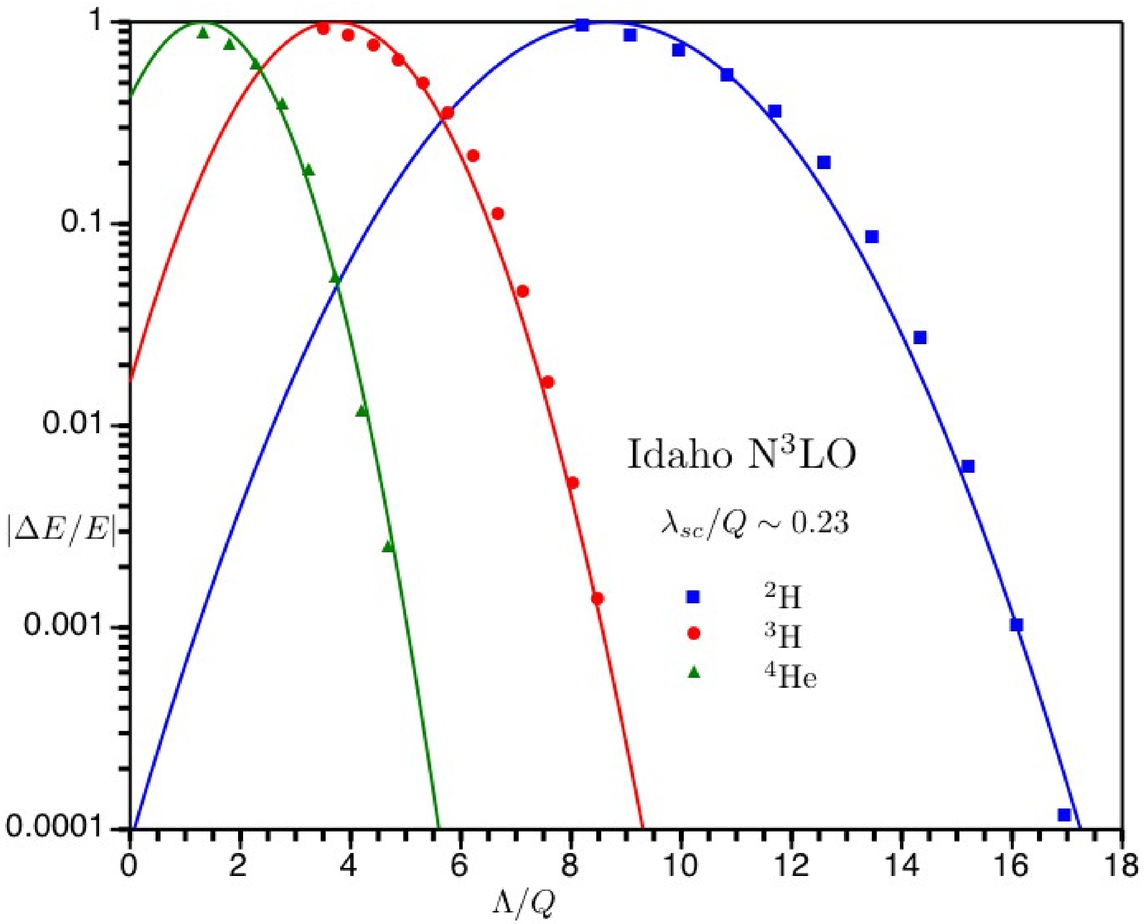




Result independent of nucleus

Universal curve at $\Lambda \leq \Lambda^{NN}$ for low λ_{sc}

Single curve for $\lambda_{sc} \leq \lambda_{sc}^{NN}$ and $\Lambda \leq \Lambda^{NN}$

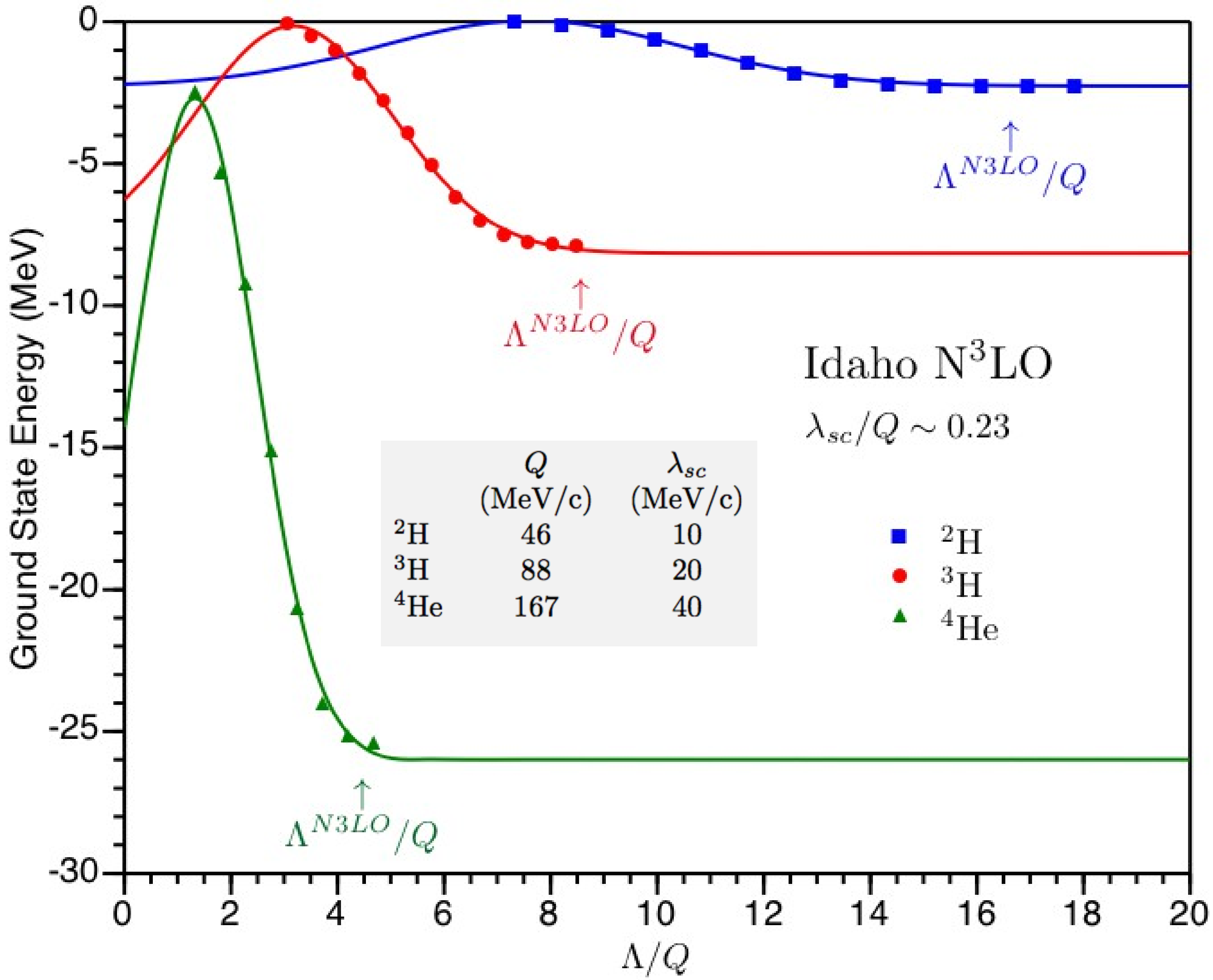


	Q (MeV/c)	λ_{sc} (MeV/c)
^{2}H	46	10
^{3}H	88	20
^{4}He	167	40

Scale each model space cutoff by binding momentum of nucleus Q to demonstrate universal (i.e., independent of particle number) nature of the Gaussian fits to low values of UV cutoff Λ

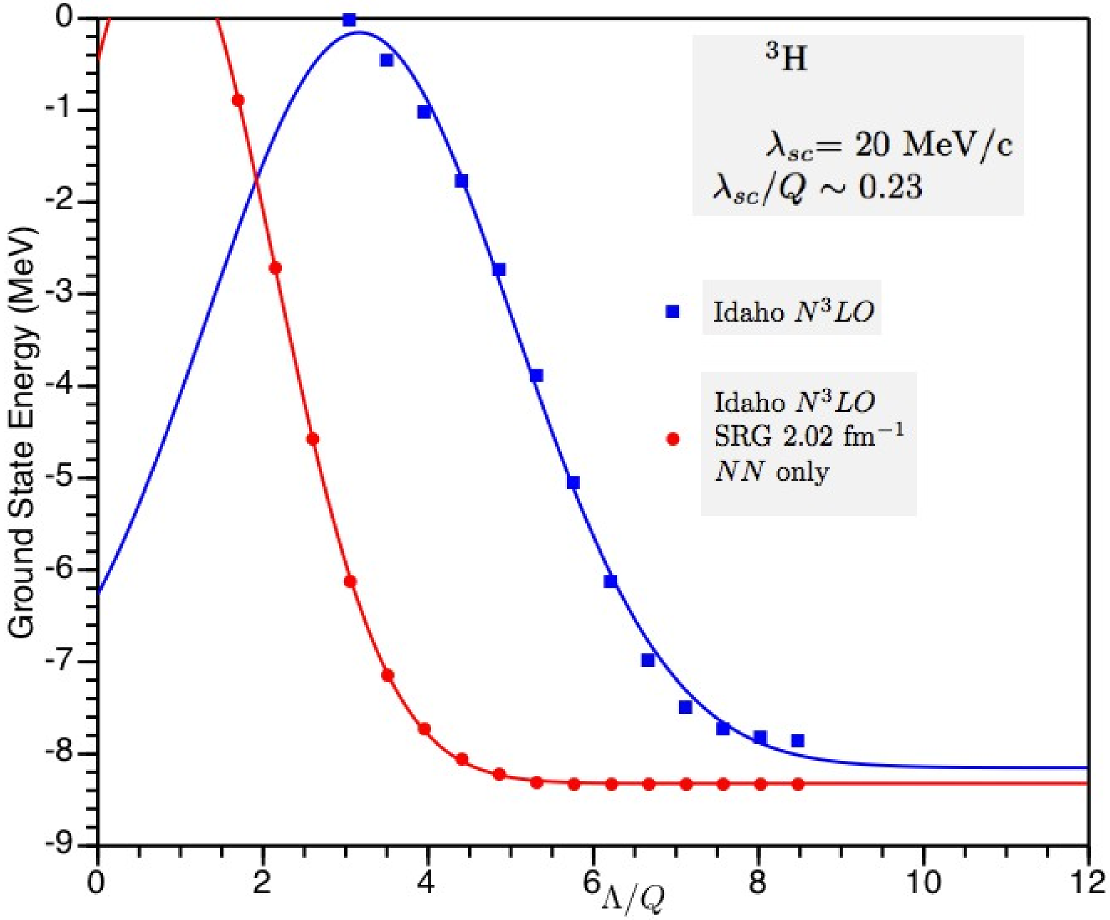
UV extrapolations with $\Lambda < \Lambda^{NN}$

$$E(\Lambda/Q) = a \exp\left(-(\Lambda/Q - b)^2/2c^2\right) + E(\Lambda/Q = \infty)$$



Extrapolated energies do NOT agree with independent calculations but are *lower*: 2 keV for deuteron, 300 keV (or 4%) for triton and 620 keV (or 2.4%) for alpha

UV extrapolations with $\Lambda < \Lambda^{NN}$

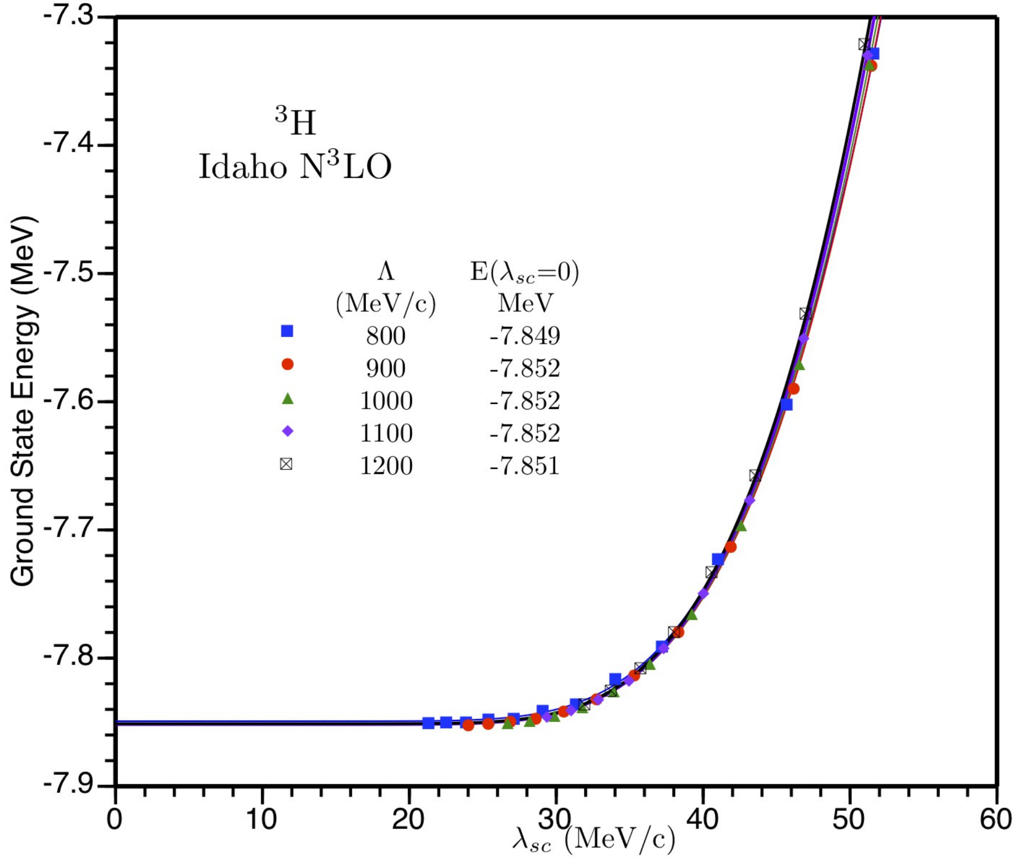


1) Extrapolation agrees with independent calculations only for SRG transformed potential.

2) Extrapolation with other values of fixed λ_{sc} is neither reliable nor robust.

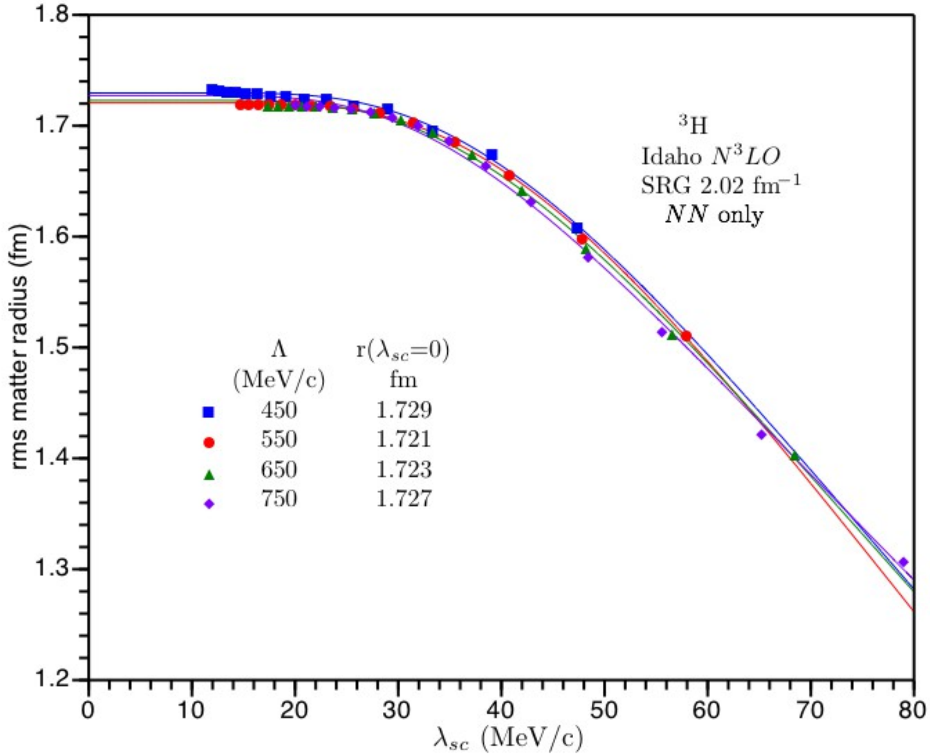
IR extrapolations with $_{sc}$

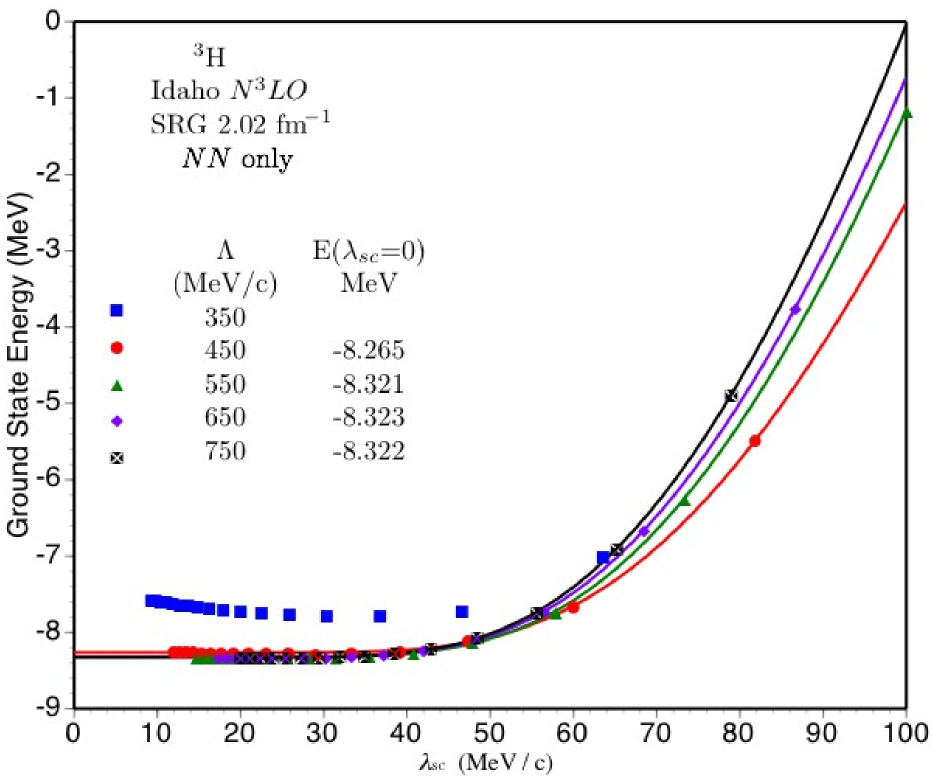
$$E(\lambda_{sc}) = A\exp(-B/\lambda_{sc}) + E(\lambda_{sc}=0)$$



If UV cutoff is large enough, all extrapolations agree with each other and with the accepted value of -7.85 MeV

IR extrapolations with λ_{sc}





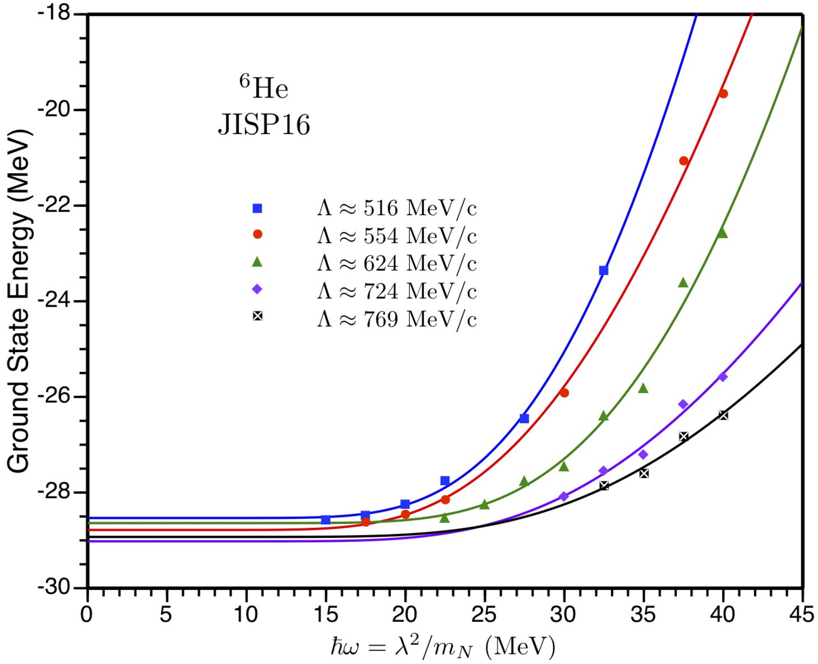
Radius extrapolation

$$r(\lambda_{sc}) = -A \exp\left(-B/\lambda_{sc}\right) + r(\lambda_{sc} = 0)$$

Energy extrapolation

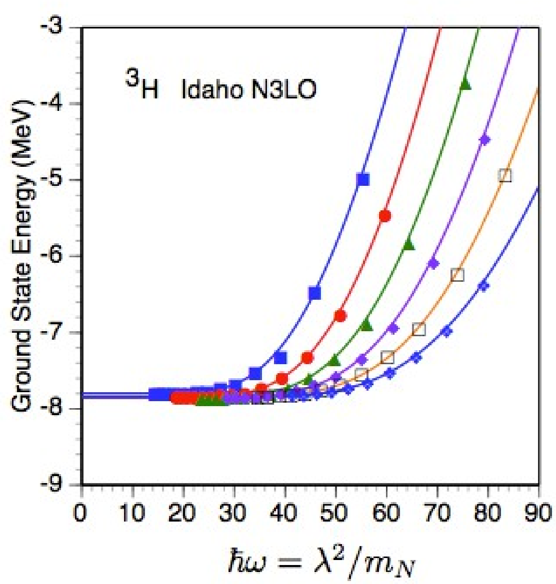
$$E(\lambda_{sc}) = A \exp(-B/\lambda_{sc}) + E(\lambda_{sc} = 0)$$

IR extrapolations with λ



We fit the ground state energy with three adjustable parameters using the relation $E_{gs}(\hbar\omega) = a\exp(-b/\hbar\omega) + E_{gs}(\hbar\omega = 0)$ five times, once for each "fixed" value of Λ. It is readily seen that one can indeed make an ir extrapolation by sending $\hbar\omega \to 0$ with fixed Λ as first advocated in Ref. [35] and that the five ir extrapolations are consistent. The spread in the five extrapolated values is about 500 keV or about 2% about the mean of -28.78 MeV. The standard deviation is 200 keV.

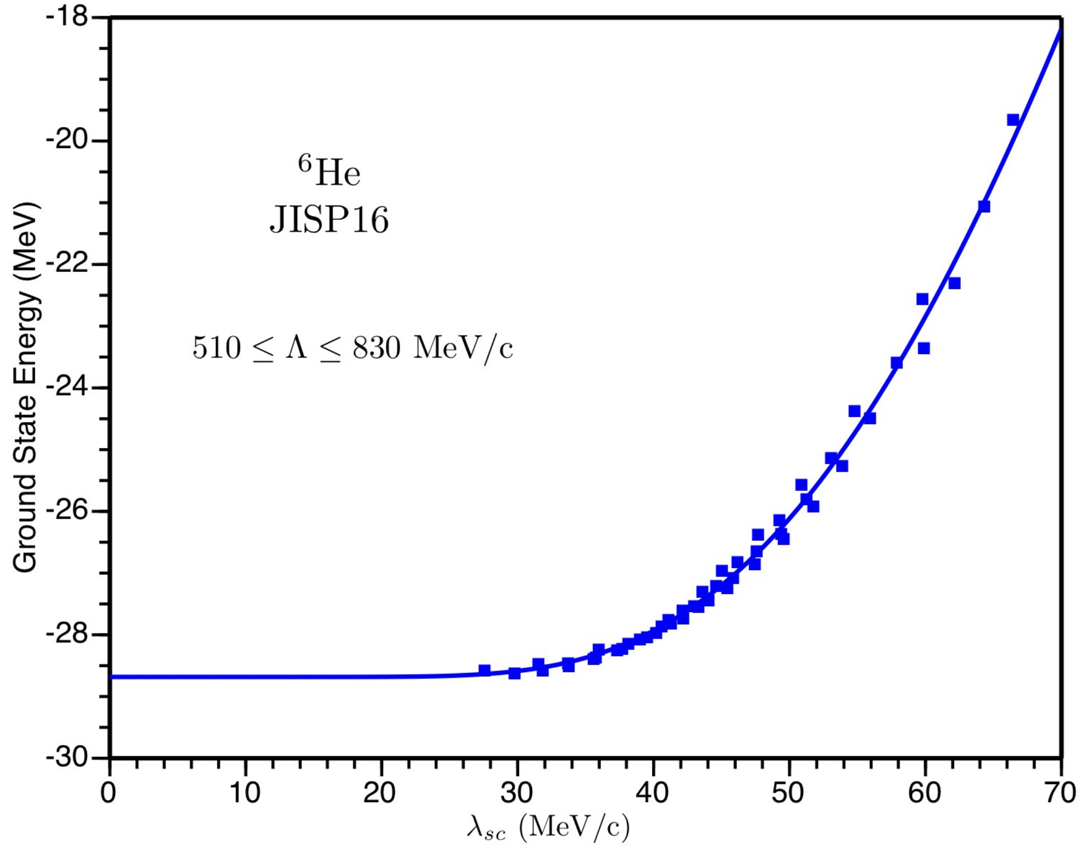
S. A. Coon, M. I. Avetian, M. K. G. Kruse, U. Van Kolck, P. Maris, J. P. Vary
Archive:1205.3230, PRC 86, 054002 (2012)



Λ =700 MeV/c E(0) = -7.798 MeV
Λ =800 eV/c E(0) = -7.844 MeV
Λ =900 MeV/c E(0) = -7.844 MeV
Λ =1000 MeV/c E(0) = -7.843 MV
Λ =1100 MeV/c E(0) = -7.841 MeV
Λ =1200 MeV/c E(0) = -7.847 MeV

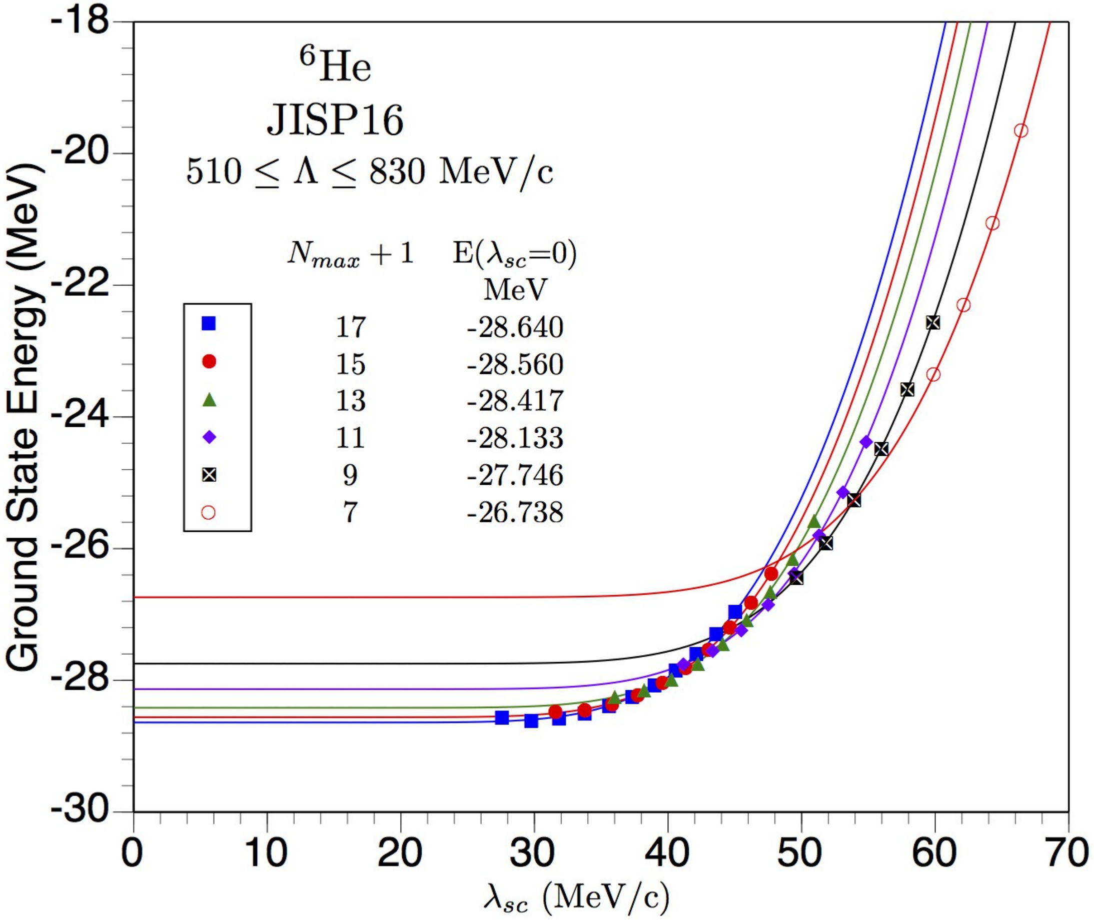
NCSM (Navratil) E= -7.85(1) MeV
Faddeev (Machleidt) E= -7.85 MeV
HH (Pisa) E= -7.854 MeV

IR extrapolations with λ_{sc}



In conclusion, our extrapolations in the ir cutoff λ of $-28.78(50)$ MeV or the ir cutoff λ_{sc} of 28.68(22) MeV are consistent with each other and with the independent calculations.

Map (N,ħω) onto (Λ,λ_{sc}) holding N fixed



Accepted extrapolation is -28.68(22) MeV

Of the 6 extrapolations only the three with N≥13 are consistent with this number.

But mean of these three large N extrapolations is -28.54 MeV with standard deviation of 0.11 MeV.

Naively concentrating on large N gives a worse extrapolation than using all points with Λ > 500 MeV/c.

Moral: results with low N can usefully stabilize and bound an extrapolation to the IR limit.

Summary

HO shell model provides a linear trial function for a variational calculation of few-body systems.

Traditional extrapolation from finite model space (N, ħω) is based upon extension of basis (N) “guided by” considerations of non-linear scale parameter (ħω).

Effective Field Theory concepts applied to a discrete basis suggest an alternative extrapolation approach based upon (Λ, λ_{sc}) which respects ultraviolet (UV) and infrared (IR) running of the results as the basis is extended.

Intrinsic UV and IR scales of the NN interaction are identified.

Extrapolation in UV with IR cutoff of model space below intrinsic IR scale is neither robust nor reliable.

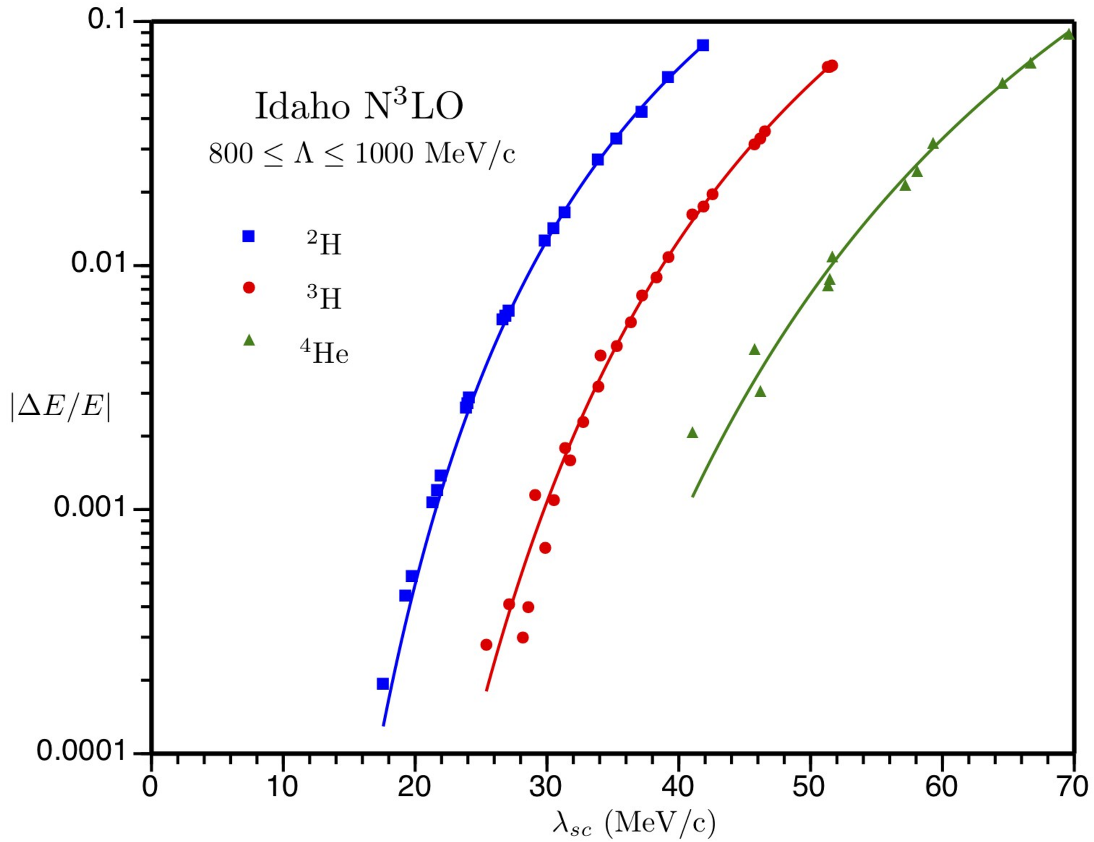
Extrapolation in IR with UV cutoff of model space above intrinsic UV scale is quite successful.

Continuing need for higher order (in λ_{sc}/Λ) corrections to these lowest order λ_{IR} and Λ_{UV}

Happy Birthday, James

Extra slides

One can use this universal scaling behavior to make an extrapolation which is independent of particle number



Data points are fit to $y = A \exp\left(-B/\lambda_{sc}\right)$

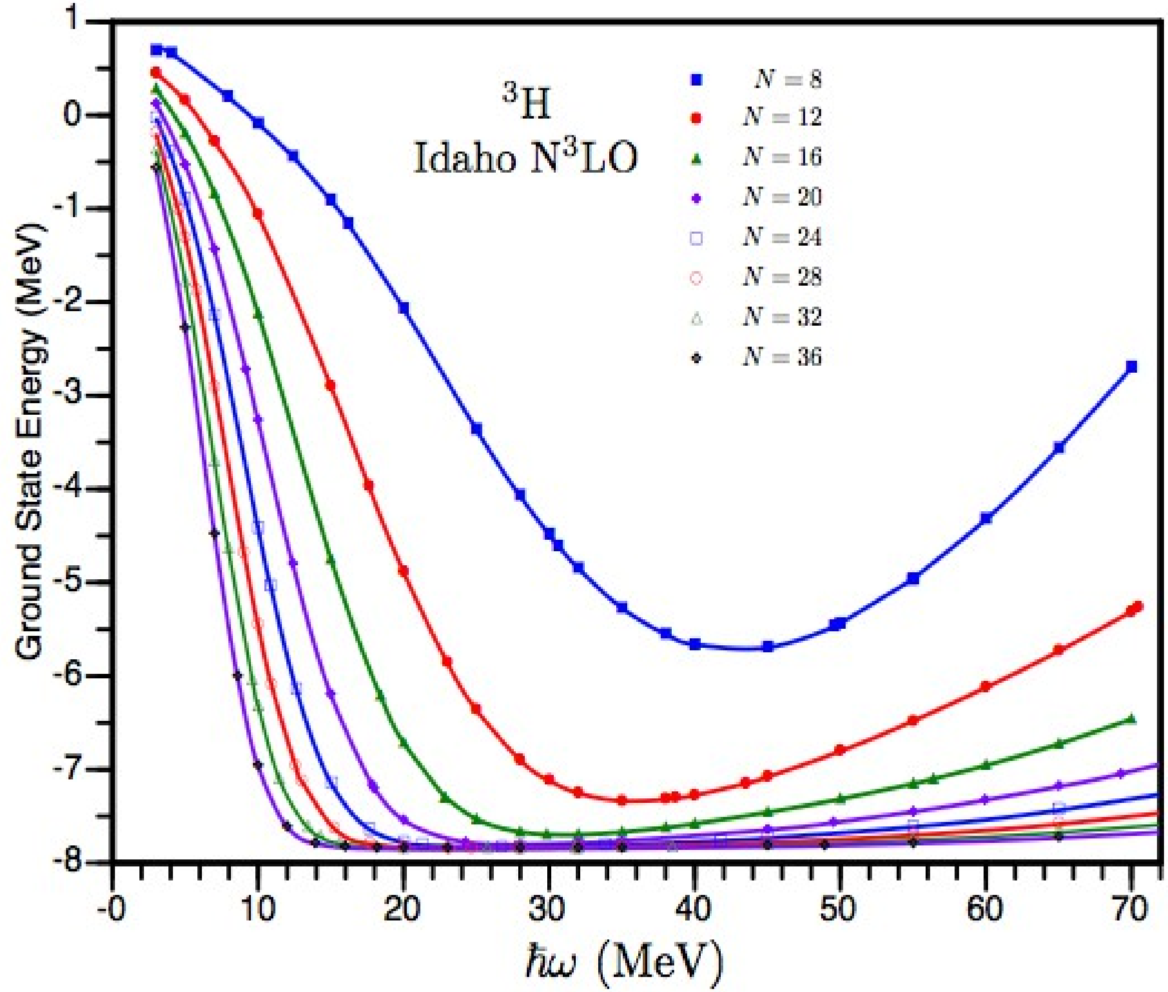


Figure 8: (Color online) Dependence of the ground-state energy of ^{3}H upon $\hbar\omega = \lambda^2/m_N = \lambda_{sc}^2/[m_N(N+3/2)]$ for fixed $N = \Lambda^2/\lambda^2 - 3/2 = \Lambda/\lambda_{sc} - 3/2$. Curves are not fits but spline interpolations to guide the eye.