Relativistic Symmetries in Hadrons and Nuclei

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Physics Reports 414 (2005) 165–261

APPROXIMATE TREATMENT OF CORRELATIONS IN NUCLEAR SPECTROSCOPY:

APPLICATIONS TO THE LEAD REGION

by

James Vary

1970

A Dissertation Presented to the Faculty of

The Graduate School of Yale University

in Candidacy for the Degree of

Doctor of Philosophy.

Pseudo-spin Symmetry in Nuclei

Almost 45 years ago a quasi-degeneracy was observed in single-nucleon doublets in nuclei with quantum numbers

$$(n, \ell, j), (n-1, \ell+2, j)$$

$$j = \tilde{\ell} \pm \tilde{s}, \quad \tilde{s} = 1/2$$

$\tilde{\ell}$ pseudo-orbital angular momentum, $\tilde{s}$ pseudospin

$2s_{1/2}$, $1d_{3/2}$ — $(\tilde{1})_{3/2, 1/2}$

$2p_{3/2}$, $1f_{5/2}$ — $(\tilde{2})_{5/2, 3/2}$

$1d_{3/2}$, $0g_{5/2}$ — $(\tilde{3})_{7/2, 5/2}$

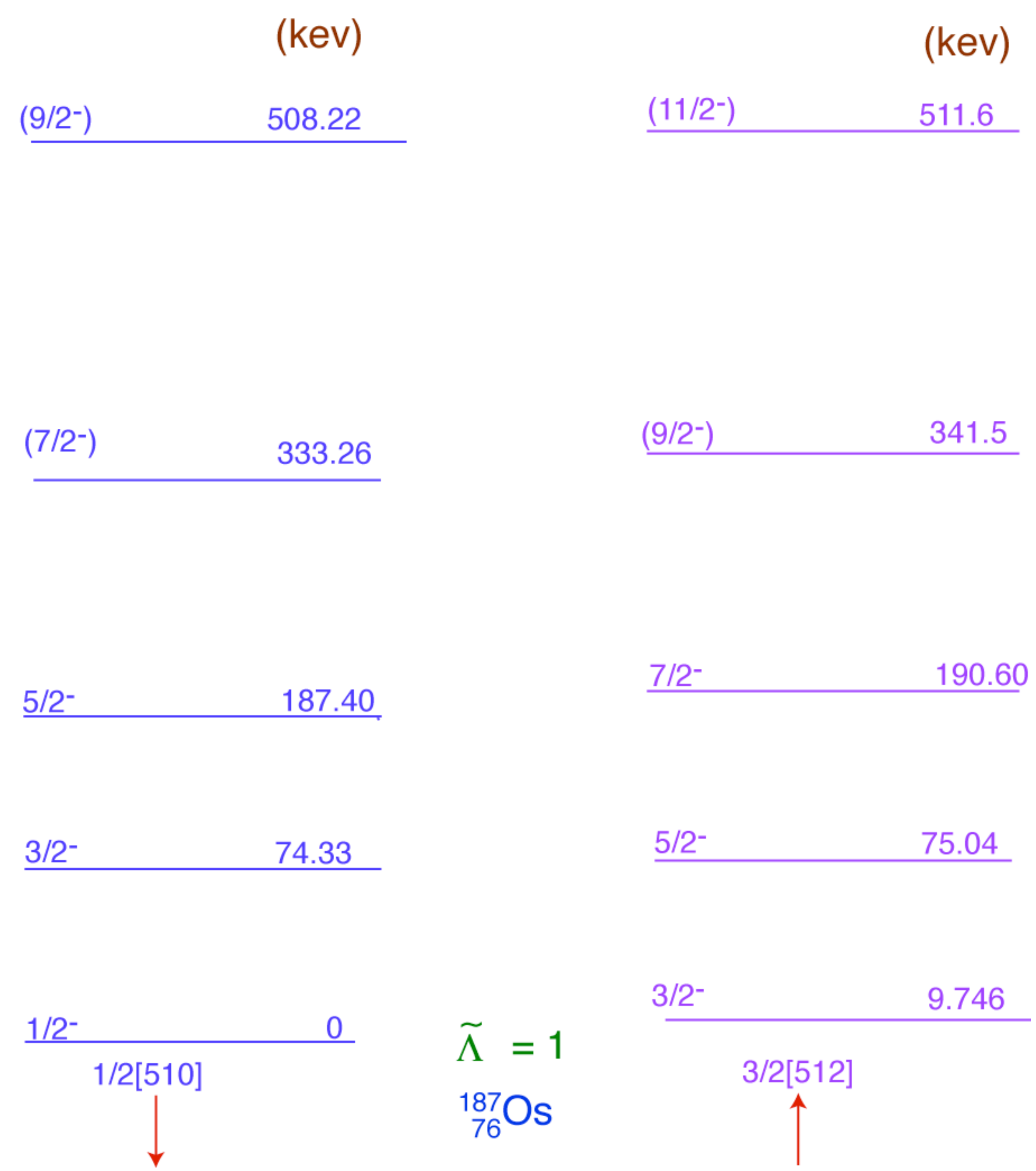
$1f_{7/2}$, $0h_{9/2}$ — $(\tilde{4})_{9/2, 7/2}$

A. Arima, M. Harvey, K. Shimizu, Phys. Lett. B 30 (1969) 517

K.T. Hecht, A. Adler, Nucl. Phys. A 137 (1969) 129.

Rotational bands built on different alignments of pseudospin along the body fixed axis.

$\Omega[N\, n_3\, \Lambda]$



A. Bohr, I. Hamamoto, B.R. Mottelson, Phys. Scr. 26 (1982) 267.

The Dirac Hamiltonian

$$H = [\vec{\alpha} \cdot \vec{p} + \beta(m + V_S(\vec{\mathbf{r}})) + V_V(\vec{\mathbf{r}})]$$

α, β are the usual Dirac matrices

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

σ_i = Pauli matrices.

Nucleons move in a scalar, $V_S(\vec{\mathbf{r}})$, and vector, $V_V(\vec{\mathbf{r}})$, mean fields.

The Dirac Hamiltonian has an invariant SU(2) symmetry for two limits:

$$V_S(\vec{r}) - V_V(\vec{r}) = C_S \quad \textit{Spin Symmetry}$$

$$V_S(\vec{r}) + V_V(\vec{r}) = C_S \quad \textit{Pseudospin Symmetry}$$

Spin Symmetry occurs in the spectrum of a:
1) meson with one heavy quark (PRL 86, 204 (2001))
2) anti-nucleon bound in a nucleus (Phys. Rep. 315, 231 (1999))

Pseudospin Symmetry occurs in the spectrum of nuclei
PRL 78, 436 (1997)

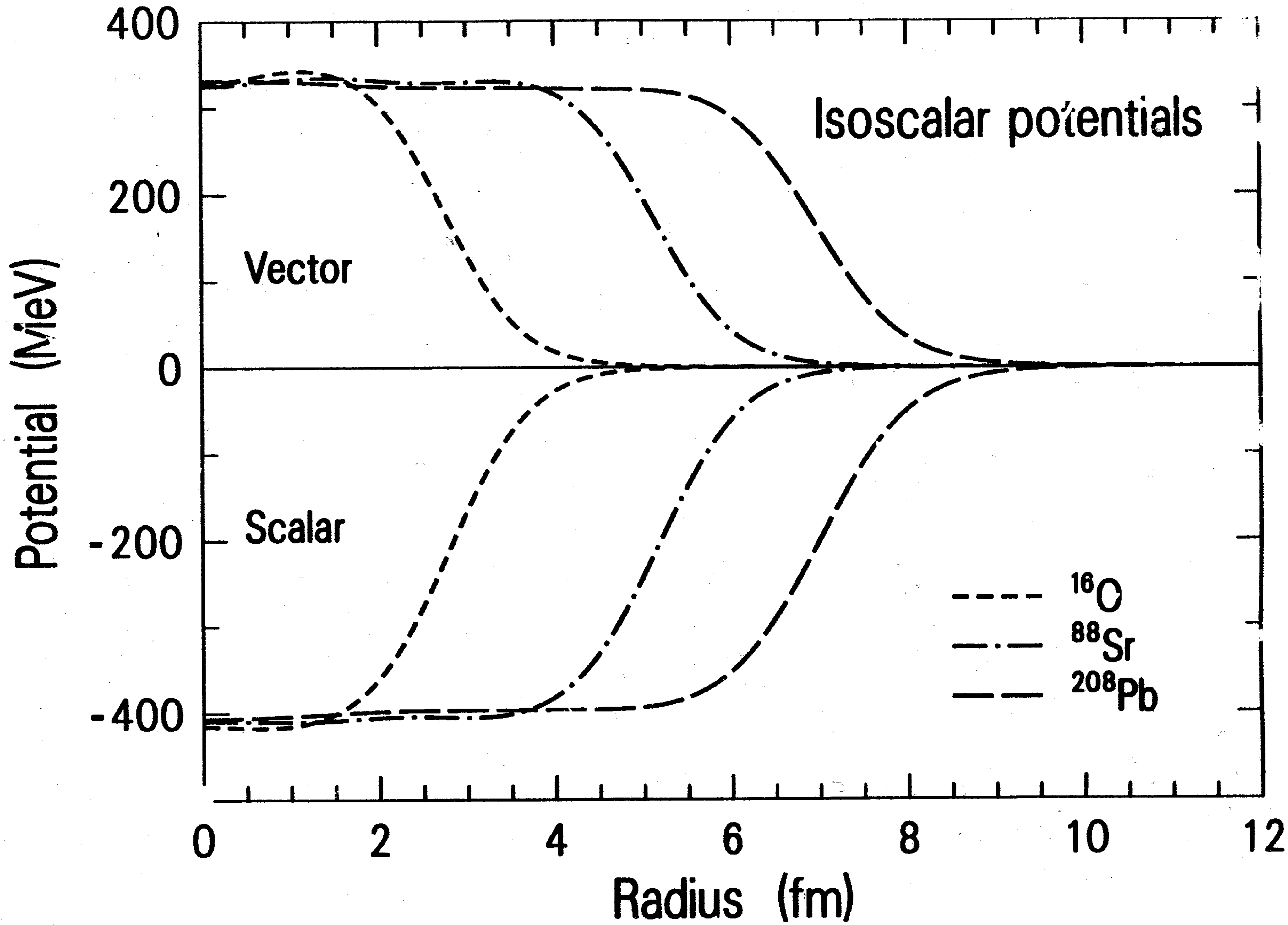

400
200
0
-200
-400
Potential (MeV)
Vector
Scalar
Isoscalar potentials
^{16}O
^{88}Sr
^{208}Pb
0
2
4
6
8
10
12
Radius (fm)

Figure 9.

QCD SUM RULES

$$\frac{V_S}{V_V} \approx -\frac{\sigma_N}{8m_q}$$

σ_N is the chiral symmetry breaking nucleon sigma term

m_q is the average quark mass

$$\sigma_N \approx 45\ MeV, m_q \approx 5\ MeV$$

$$\frac{V_S}{V_V} \approx -1.1$$

Uncannily close to the ratio of central values of mean field potentials

T.D. Cohen, R.J. Furnstahl, D.K. Griegel, X. Jin, Prog. Part. Nucl. Phys. 35 (1995) 221.

Pseudospin Generators

$$\vec{\tilde{S}} = \begin{pmatrix} \vec{\tilde{s}} & 0 \\ 0 & \vec{s} \end{pmatrix}$$

$$\vec{s} = \vec{\sigma}/2 \qquad \vec{\tilde{s}} = U_p \vec{s} U_p \qquad U_p = \vec{\sigma} \cdot \hat{p}$$

These pseudo-spin generators commute with the Dirac Hamiltonian in the pseudo-spin limit independent of the form of the potentials, spherical, deformed, triaxial:

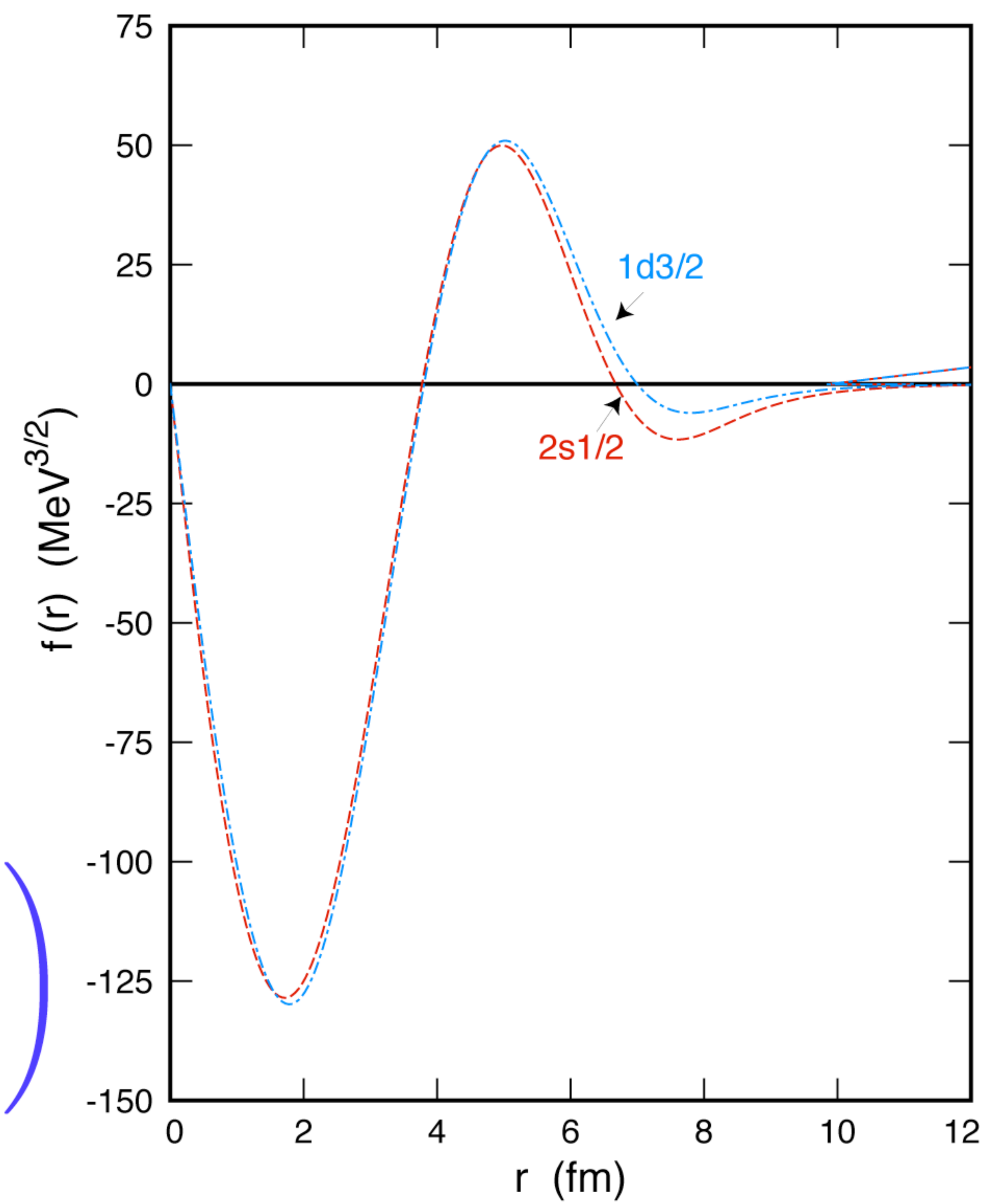
$$V_S(\vec{r}) = -V_V(\vec{r}) + C_{ps}$$

and have spin-like commutation relations

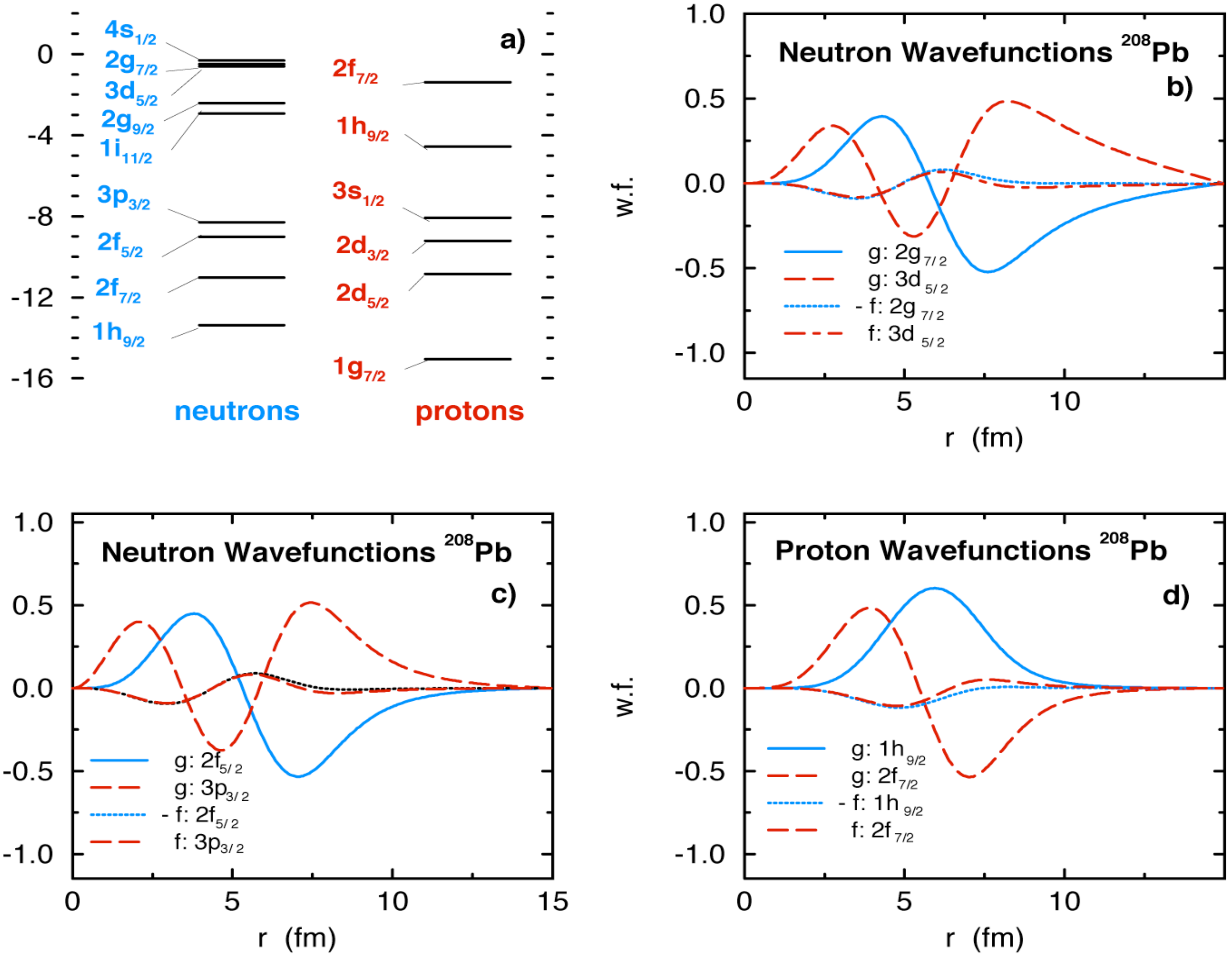
$$[H_{ps}, \vec{\tilde{S}}_i] = 0 \qquad [\vec{\tilde{S}}_i, \vec{\tilde{S}}_j] = i\varepsilon_{ijk}\vec{\tilde{S}}_k$$

Pseudospin Symmetry predicts that the lower spatial amplitudes of the two eigenstates in the doublets are equal

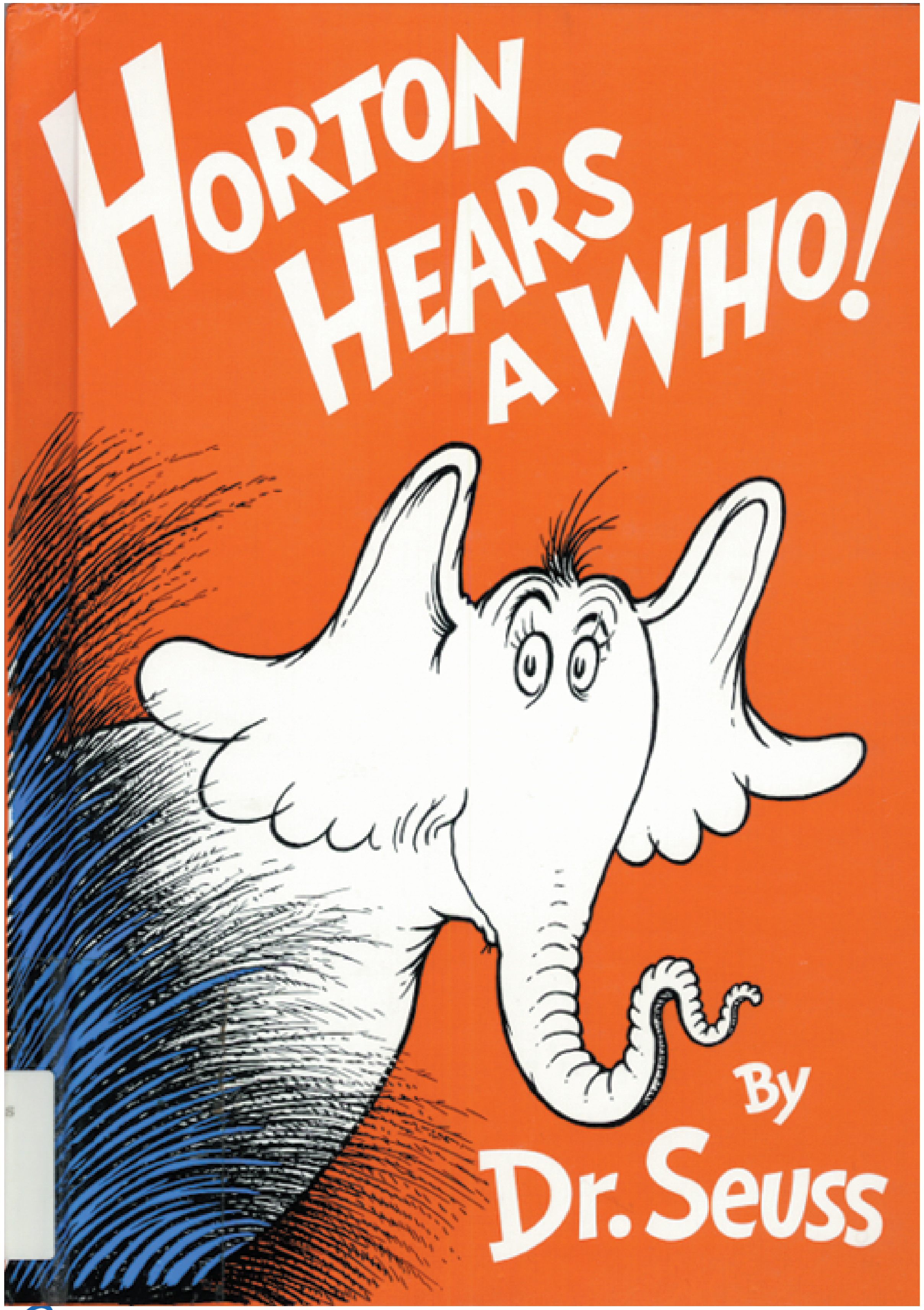
$$\hat{S}_i = \begin{pmatrix} \hat{\tilde{s}}_i & 0 \\ 0 & \hat{s}_i \end{pmatrix}$$



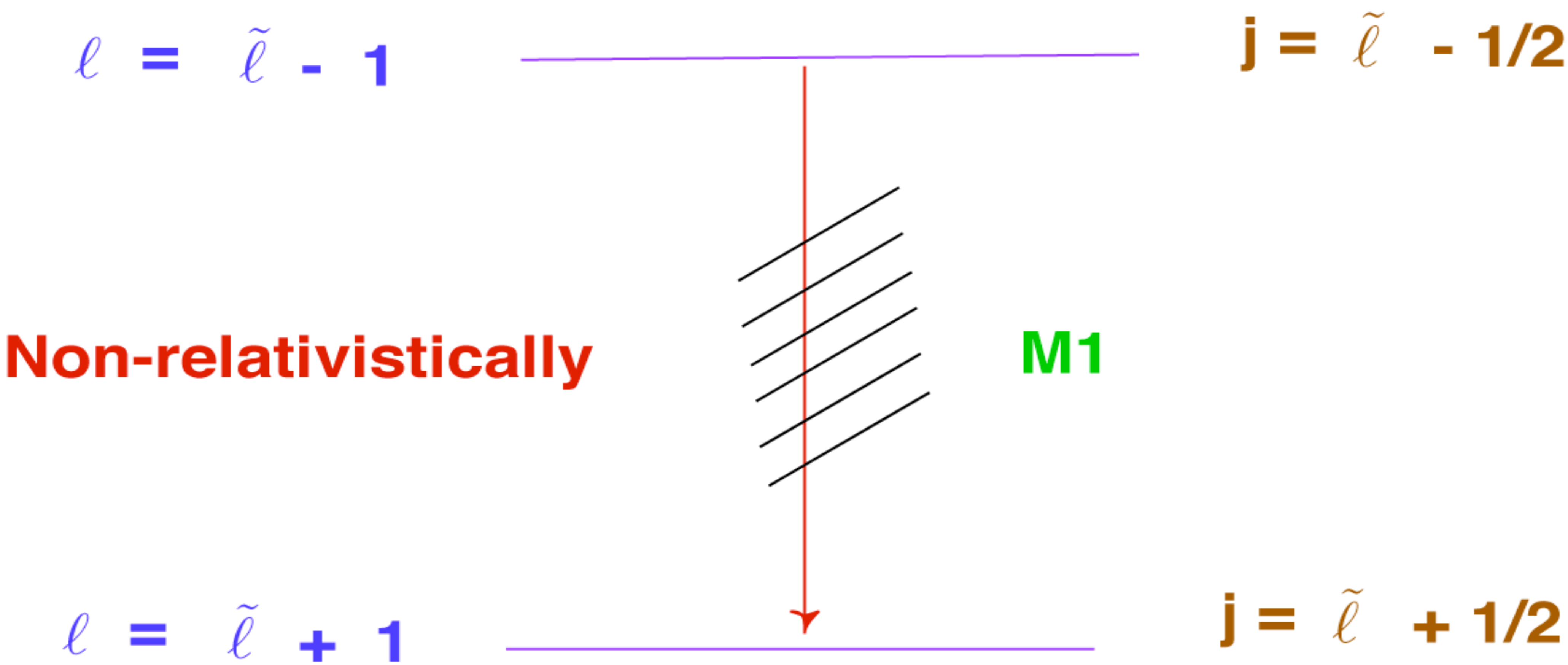
Upper (g) and Lower (f) Radial Wavefunctions



J. Meng, K. Sugawara-Tanabe, S. Yamaji, P. Ring, A. Arima, Phys. Rev. 58 (1998) R628.


HORTON HEARS A WHO!
By
Dr. Seuss

Magnetic Transitions between Pseudospin doublets



Magnetic Transitions for Pseudospin Symmetry

$$B(M1: j' \rightarrow j\,)_\nu = \frac{j+1}{2j+1}\left[\mu_{j,\nu} - \mu_{A,\nu}\right]^2 \qquad j' = \tilde{\ell} + 1/2$$

$$B(M1: j' \rightarrow j\,)_\nu = \frac{2j+1}{j+1}\left[\frac{j+2}{2j+3}\left(\mu_{j',\nu} + \frac{j+1}{j+2}\mu_{A,\nu}\right)\right]^2 \qquad j = \tilde{\ell} - 1/2$$

Magnetic Transitions for Pseudospin Symmetry

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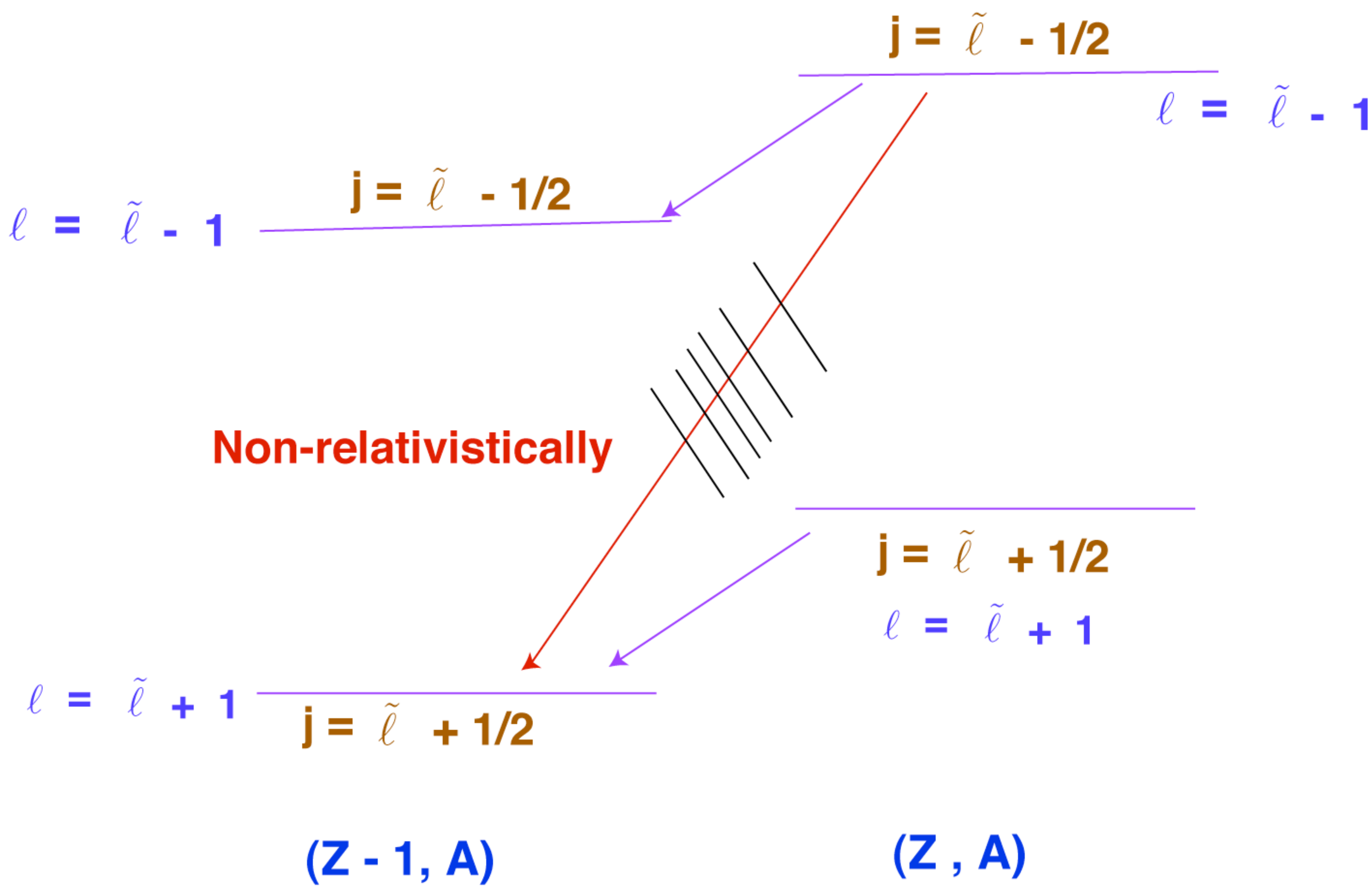
Predicted "ℓ forbidden" magnetic dipole transition in ^{39}Ca.

	$B(M1 : j,\nu \rightarrow j',\nu)$
Predicted Equation	0.0166
Predicted Equation	0.0121
EXP	0.0121 (14)

Gamow-Teller Transitions between Pseudospin doublets



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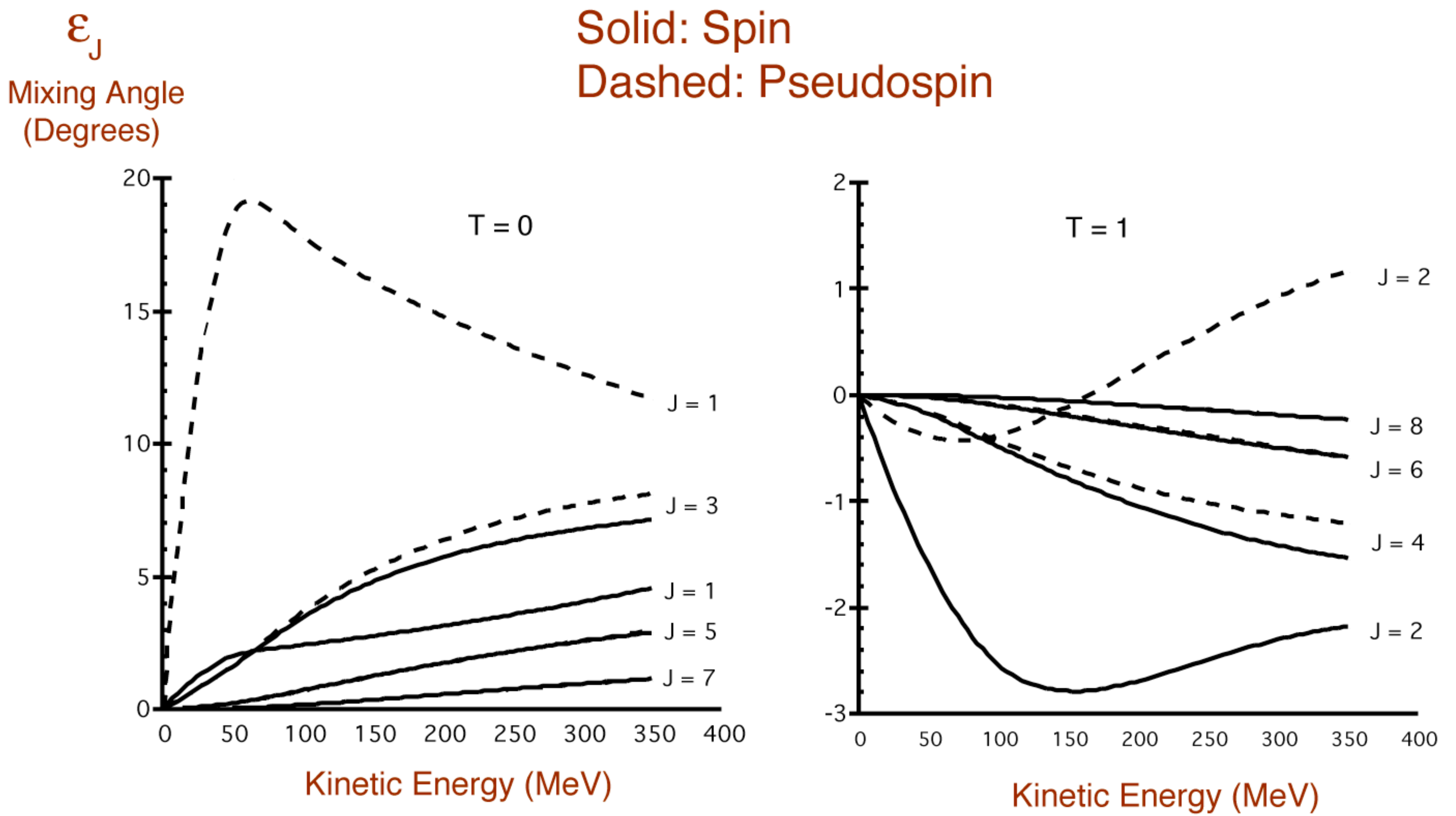
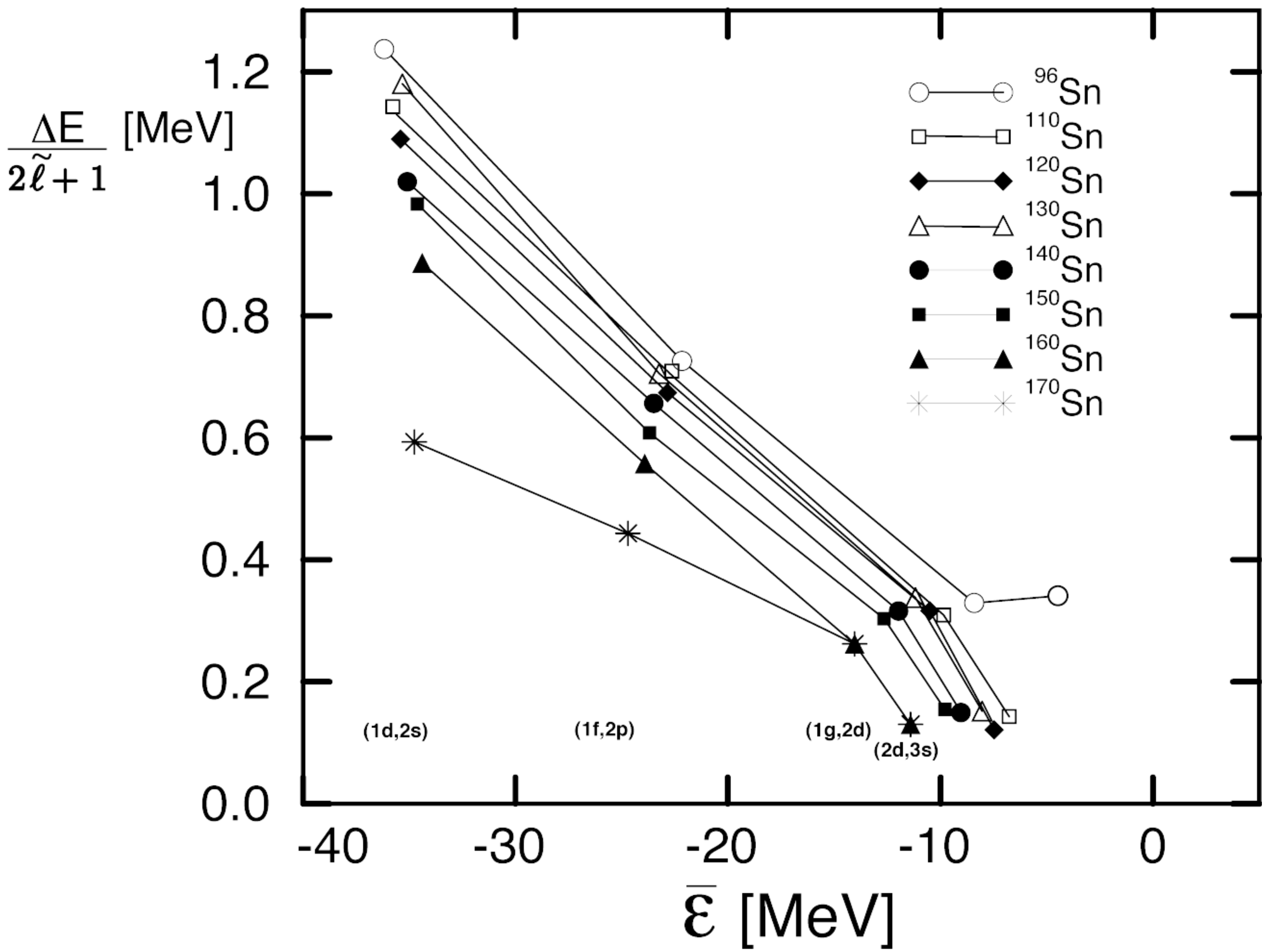
Nucleon-nucleon scattering

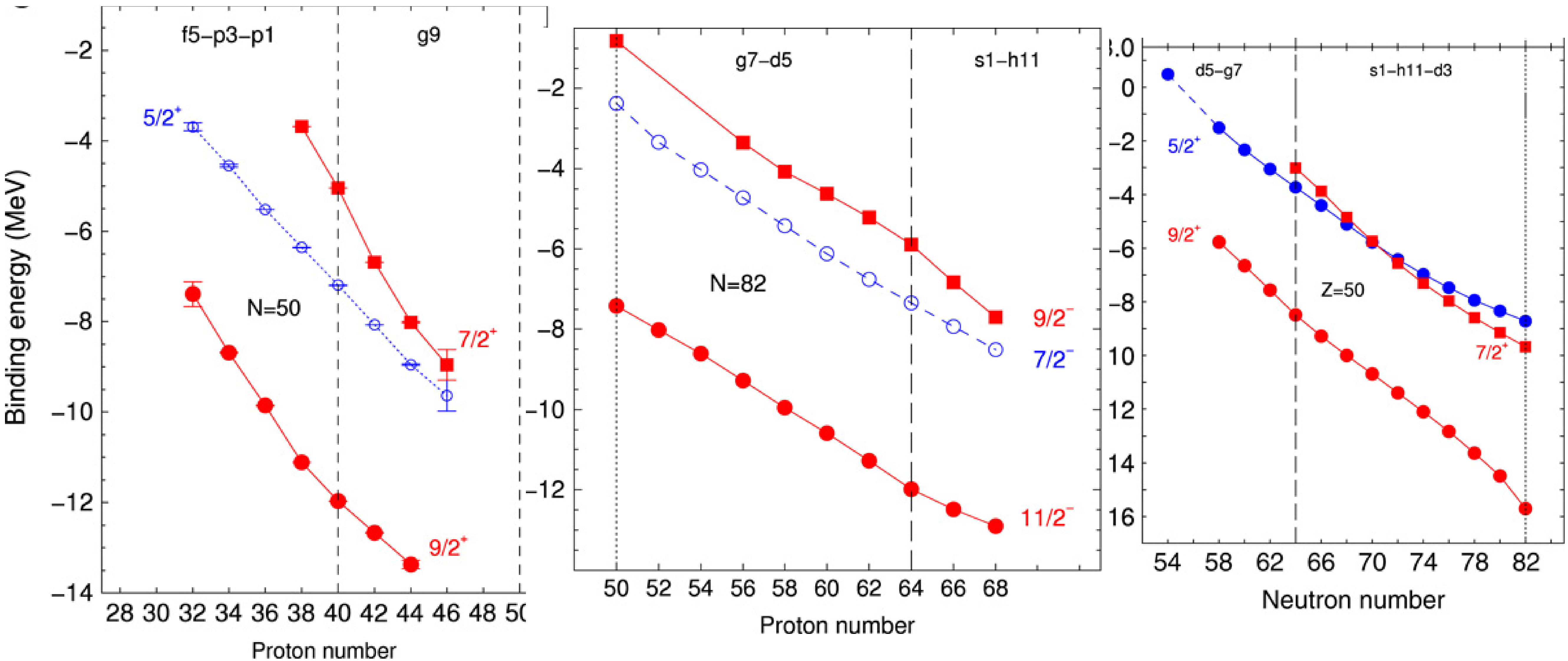
Mixing Angles



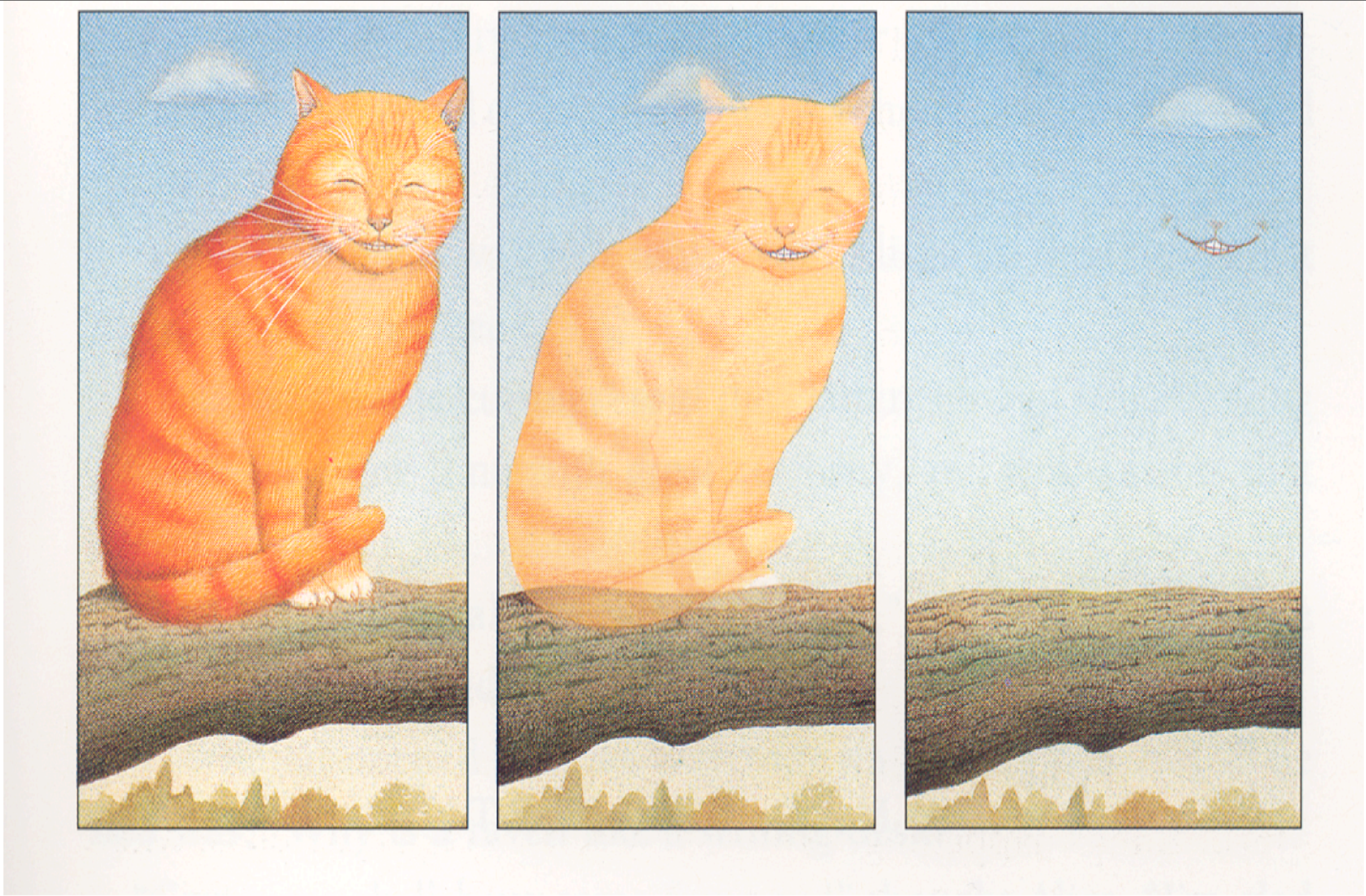


The pseudospin energy splitting as a function of the average energy of the doublets. Symmetry improves closer to the Fermi sea and with neutron excess.

J. Meng, K. Sugawara-Tanabe, S. Yamaji, A. Arima, Phys. Rev. C 59 (1999) 154.



Progress in Particle and Nuclear Physics 61 (2008) 602–673



“Well! I’ve often seen a cat without a grin,” thought Alice, “but a grin without a cat! It’s the most curious thing I ever saw in all my life!”
Lewis Carroll, *Alice’s Adventures in Wonderland*

Anti-nucleon Spectrum

Charge Conjugation

$$\bar{V}_S(\vec{r}) = C^\dagger V_S(\vec{r}) C = V_S(\vec{r})$$

$$\bar{V}_V(\vec{r}) = C^\dagger V_V(\vec{r}) C = -V_V(\vec{r})$$

$$\bar{V}_S(\vec{r}) \approx \bar{V}_V(\vec{r})$$

Anti-nucleon potential will have absorption potential as well. However, the vector and scalar absorption potentials must be equal so that under charge conjugation the sum of the them will be zero for nucleons. Therefore, the absorption potential will conserve spin symmetry as well.

Spin polarization in antiproton scattering from Carbon is almost zero supporting this prediction, but data set is limited (NPA 487, 563 (1988)).

Perhaps additional antiproton scattering will be forthcoming at GSI

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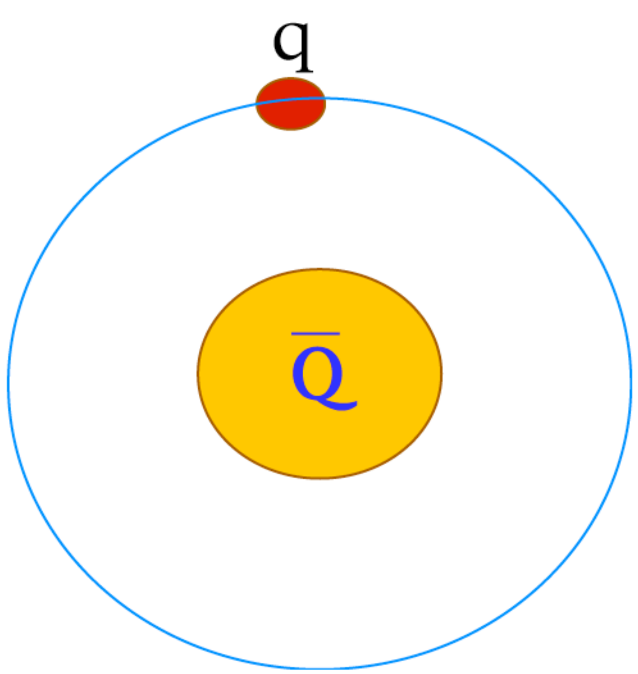
Dirac Hamiltonian has a spin symmetry if

$$V_S(x) = V_V(x) + C_S$$

Explains spin degeneracies of Light Quark - Heavy Quark Mesons



Phys. Rev. L ett. 86, 204 (2001); hep-ph/0002094

Beyond the Relativistic Mean Field:

Non Relativistic Shell Model Hamiltonians that have Pseudo-spin and Pseudo-Angular Momentum as Dynamic Symmetries

$$H = \sum_{k=1}^{A} h(\vec{p}_k) + \sum_{k\leq t}^{A} V(\vec{p}_k, \vec{p}_t),$$

$$\vec{\tilde{s}}_k = U_p \; \vec{s}_k \; U_p = 2 \; \vec{s}_k \cdot \hat{p} \; \hat{p}_k - \vec{s}_k$$

$$\vec{\tilde{\ell}}_k = U_{p_k} \; \vec{\ell}_k \; U_{p_k} = -2 \; s_k \cdot \hat{p} \; \hat{p}_k + \vec{\ell}_k + 2\vec{s}_k$$

$$V(\vec{p}_k, \vec{p}_t) =$$

$$(\tilde{V}_c^{(0)}(p_k, p_t, \theta_{k,t}) + \tilde{V}_{ps}^{(0)}(p_k, p_t, \theta_{k,t})\tilde{s}_k \cdot \tilde{s}_t + \tilde{V}_{po}^{(0)}(p_k, p_t, \theta_{k,t})\tilde{\ell}_k \cdot \tilde{\ell}_t + \tilde{V}_{pso}^{(0)}(p_k, p_t, \theta_{k,t})(\tilde{s}_k \cdot \tilde{\ell}_t + \tilde{s}_t \cdot \tilde{\ell}_k))\frac{(1-\tau_k \cdot \tau_t}{4}$$

$$+(\tilde{V}_c^{(1)}(p_k, p_t, \theta_{k,t}) + \tilde{V}_{ps}^{(1)}(p_k, p_t, \theta_{k,t})\tilde{s}_k \cdot \tilde{s}_t + \tilde{V}_{po}^{(1)}(p_k, p_t, \theta_{k,t})\tilde{\ell}_k \cdot \tilde{\ell}_t + \tilde{V}_{pso}^{(1)}(p_k, p_t, \theta_{k,t})(\tilde{s}_k \cdot \tilde{\ell}_t + \tilde{s}_t \cdot \tilde{\ell}_k))\frac{(3+\tau_k \cdot \tau_t}{4}$$

$$V^{(T)}(\vec{p}_k, \vec{p}_t) =$$

$$V_c^{(T)}(p_k, p_t, \theta_{k,t}) + V_s^{(T)}(p_k, p_t, \theta_{k,t}) s_k \cdot s_t + V_o^{(T)}(p_k, p_t, \theta_{k,t}) \ell_k \cdot \ell_t + V_{so}^{(T)}(p_k, p_t, \theta_{k,t})(s_k \cdot \ell_t + s_t \cdot \ell_k)$$

$$V^{(T)}(\vec{p}_k, \vec{p}_t) =$$

$$V_c^{(T)}(p_k, p_t, \theta_{k,t}) + V_s^{(T)}(p_k, p_t, \theta_{k,t}) s_k \cdot s_t + V_o^{(T)}(p_k, p_t, \theta_{k,t}) \ell_k \cdot \ell_t + V_{so}^{(T)}(p_k, p_t, \theta_{k,t})(s_k \cdot \ell_t + s_t \cdot \ell_k)$$

$$+V_t^{(T)}(p_k, p_t, \theta_{k,t})[s_k s_t]^{(2)} \cdot ([\hat{p}_k \hat{p}_k]^{(2)} + [\hat{p}_t \hat{p}_t]^{(2)})$$

$$+ V_{dt}^{(T)}(p_k, p_t, \theta_{k,t})([s_k s_t]^{(2)} \cdot [\hat{p}_k \hat{p}_t]^{(2)} - [s_k s_t]^{(1)} \cdot [\hat{p}_k \hat{p}_t]^{(1)})$$

$$+V_{mso}^{(T)}(p_k, p_t, \theta_{k,t})([s_k \cdot \hat{p}_k]\hat{p}_k \cdot \vec{\ell}_t + [s_t \cdot \hat{p}_t]\hat{p}_t \cdot \vec{\ell}_k)$$

In addition to spin and orbital angular momentum interactions, tensor, dipole and momentum dependent spin-orbit interactions.

T. Otsuka and D. Abe, Prog. Part. Nucl. Phys. **59**, 425 (2007).

Relativistic Harmonic Oscillator

$$V_S(\vec{r}) = \frac{M}{2}\sum_{i=1}^{3} \omega_{S,i}^2 x_i^2$$

$$V_V(\vec{r}) = \frac{M}{2}\sum_{i=1}^{3} \omega_{V,i}^2 \, x_i^2$$

In the symmetry limits the eigenfunctions and eigen-energies can be solved analytically. The eigenfunctions are similar to the non relativistic limit. That is, the upper and lower components can be written in terms of Gaussians and Laguerre polynomials. This true for spherical and non-spherical harmonic oscillator.

JNG, PRC 69, 034318 (2004)

Energy Eigenvalues in the Spin Symmetry and Spherical Symmetry Limit

$$\omega_S = \omega_V = \omega$$

$$E_N = M\left[B(A_N) + \frac{1}{3} + \frac{4}{9\,B(A_N)} \right]$$

$$B(A_N) = \left[\frac{A_N + \sqrt{A_N^2 - \frac{32}{27}}}{2} \right]^{\frac{2}{3}}$$

$$A_N = C\,(N + \frac{3}{2}), C = \frac{\sqrt{2}\,\omega}{M},$$

$$N = 2n + \ell = 0, 1, \ldots$$

Energy Eigenvalues in the Pseudo-spin Symmetry and Spherical Symmetry Limit

$$\omega_V = -\omega_S = \widetilde{\omega}$$

$$E_{\widetilde{N}} = M\left[B(A_{\widetilde{N}}) + \frac{1}{3} + \frac{4}{9\, B(A_{\widetilde{N}})} \right]$$

$$B(A_{\widetilde{N}}) = \left[\frac{A_{\widetilde{N}} + \sqrt{A_{\widetilde{N}}^2 - \frac{32}{27}}}{2} \right]^{\frac{2}{3}}$$

$$A_{\widetilde{N}} = C\,(\widetilde{N} + \frac{3}{2}), C = \frac{\sqrt{2}\,\widetilde{\omega}}{M},$$

$$\widetilde{N} = 2\tilde{n} + \tilde{\ell} = 0,1,\ldots.$$

Spherical Relativistic Harmonic Oscillator U(3) and pseudo-U(3)

In these symmetry limits the energy depends only on N or $\tilde{N}$, as in the non-relativistic harmonic oscillator. In the non-relativistic case this is because the Hamiltonian has an U(3) symmetry.

Although the energy spectrum dependence on N is different than in the non-relativistic case, the Dirac Hamiltonian for the spherical harmonic oscillator has been shown to have an U(3) symmetry for the spin symmetry limit and a pseudo-U(3) in the pseudosin limit.

JNG, PRL 95, 252501 (2005)

Generators of Spin and U(3) Symmetry

$$\vec{S} = \begin{pmatrix} \vec{s} & 0 \\ 0 & U_p \, \vec{s} \, U_p \end{pmatrix}, \vec{L} = \begin{pmatrix} \vec{\ell} & 0 \\ 0 & U_p \, \vec{\ell} \, U_p \end{pmatrix}$$

$$\vec{s} = \vec{\sigma} / 2, \;\; \vec{\ell} = \frac{(\vec{r} \times \vec{p})}{\hbar}, \;\; U_p = \frac{\vec{\sigma} \cdot \vec{p}}{p}$$

Quadrupole Generator

$$Q_m = \lambda_2 \begin{pmatrix} M\omega^2 (M\omega^2 r^2 + 2M)[rr]_m^{(2)} + [pp]_m^{(2)} & M\omega^2 [rr]_m^{(2)} \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p}\, M\omega^2 [rr]_m^{(2)} & [pp]_m^{(2)} \end{pmatrix}$$

where the norm is

$$\lambda_2 = \sqrt{\frac{3}{M\omega^2\hbar^2(H+M)}}$$

JNG, PRL 95, 252501 (2005)

The Generator that counts the number of oscillator quanta is

$$\hat{N} = \frac{\sqrt{H+M}\,(H-M)}{\hbar\sqrt{2M\omega^2}} - \frac{3}{2}$$

Pseudo U(3) Generators

$$\vec{\tilde{S}} = \begin{pmatrix} U_p\, \vec{s}\, U_p & 0 \\ 0 & \vec{s} \end{pmatrix}, \vec{\tilde{L}} = \begin{pmatrix} U_p\, \vec{\ell}\, U_p & 0 \\ 0 & \vec{\ell} \end{pmatrix},$$

$$\tilde{Q}_m = \sqrt{\frac{3}{M\omega^2\hbar^2(\tilde{H}-M)}} \begin{pmatrix} [pp]_m^{(2)} & \vec{\sigma}\cdot\vec{p}\, M\omega^2\, [rr]_m^{(2)} \\ M\omega^2\, [rr]_m^{(2)}\, \vec{\sigma}\cdot\vec{p} & M\omega^2\, (M\omega^2\, r^2 - 2M)[rr]_m^{(2)} + [pp]_m^{(2)} \end{pmatrix}$$

$$\widetilde{N} = \frac{\sqrt{H-M}\,(H+M)}{2\hbar\sqrt{M\omega^2}} - \frac{3}{2}.$$

Relativistic Harmonic Oscillator with no symmetries: $V_S(\vec{r}) \neq \pm V_V(\vec{r})$

The pseudo-spin limit is approximately valid for nuclei. However in this limit there are no bound Dirac valence states, only bound Dirac hole states. Therefore we would like to solve analytically the Dirac Hamiltonian for general scalar and vector harmonic oscillator potentials to obtain eigenfunctions with a realistic spectrum relevant for nuclei.

Furthermore, this is an interesting problem because there exists a symmetry limit for which perturbation theory is not valid even though

$$V_S(\vec{r}) \approx -V_V(\vec{r})$$

as is the case for nuclei.

Summary

- Dirac Hamiltonian has an SU(2) symmetry for

$$V_S(\vec{r}) - V_V(\vec{r}) = C_s \quad spin \ \ symmetry$$

$$V_S(\vec{r}) + V_V(\vec{r}) = C_{ps} \quad pseudo-spin \quad symmetry$$

- Nuclei exhibit pseudo-spin symmetry which may improve as isospin increases
- Predicts that anti-nucleons in a nuclear environment exhibit spin symmetry
- Mesons and Baryons seem to exhibit spin symmetry

Summary

- The relativistic harmonic oscillator has a spin and U(3) symmetry when the scalar and vector potentials are equal. This limit is relevant for hadrons and anti-nucleons in a nuclear environment and perturbation theory is possible.
- The relativistic harmonic oscillator has a pseudo-spin and a pseudo U(3) symmetry when the scalar and vector potentials are equal in magnitude and opposite in sign. This limit is relevant for nuclei, but, in the exact limit, there are no bound Dirac valence states so perturbation is not an option.
- Therefore we are attempting to solve analytically the relativistic harmonic oscillator with arbitrary vector and scalar potentials.

Future

More fundamental rationale for pseudo-spin symmetry

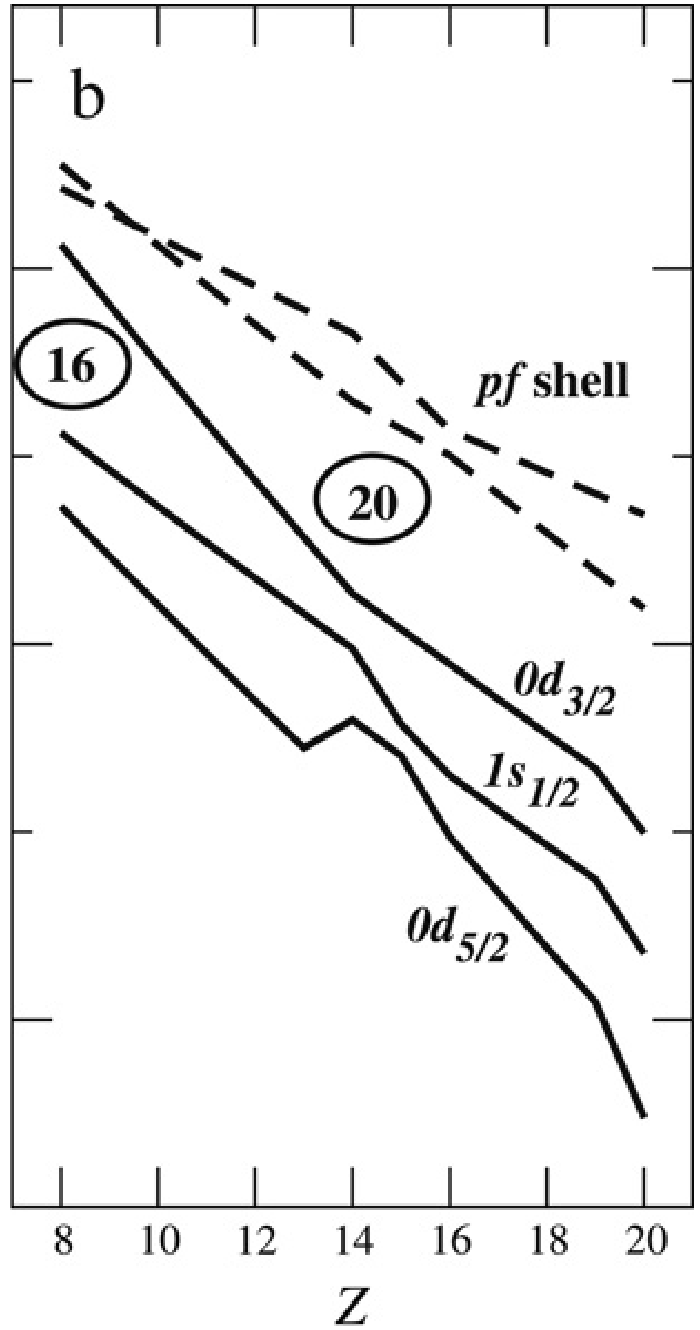
1) What is the connection between QCD and pseudo-spin symmetry suggested by QCD sum rules?

2) Does Chiral Effective Field Theory produce an interaction that approximately conserve pseudo-spin?

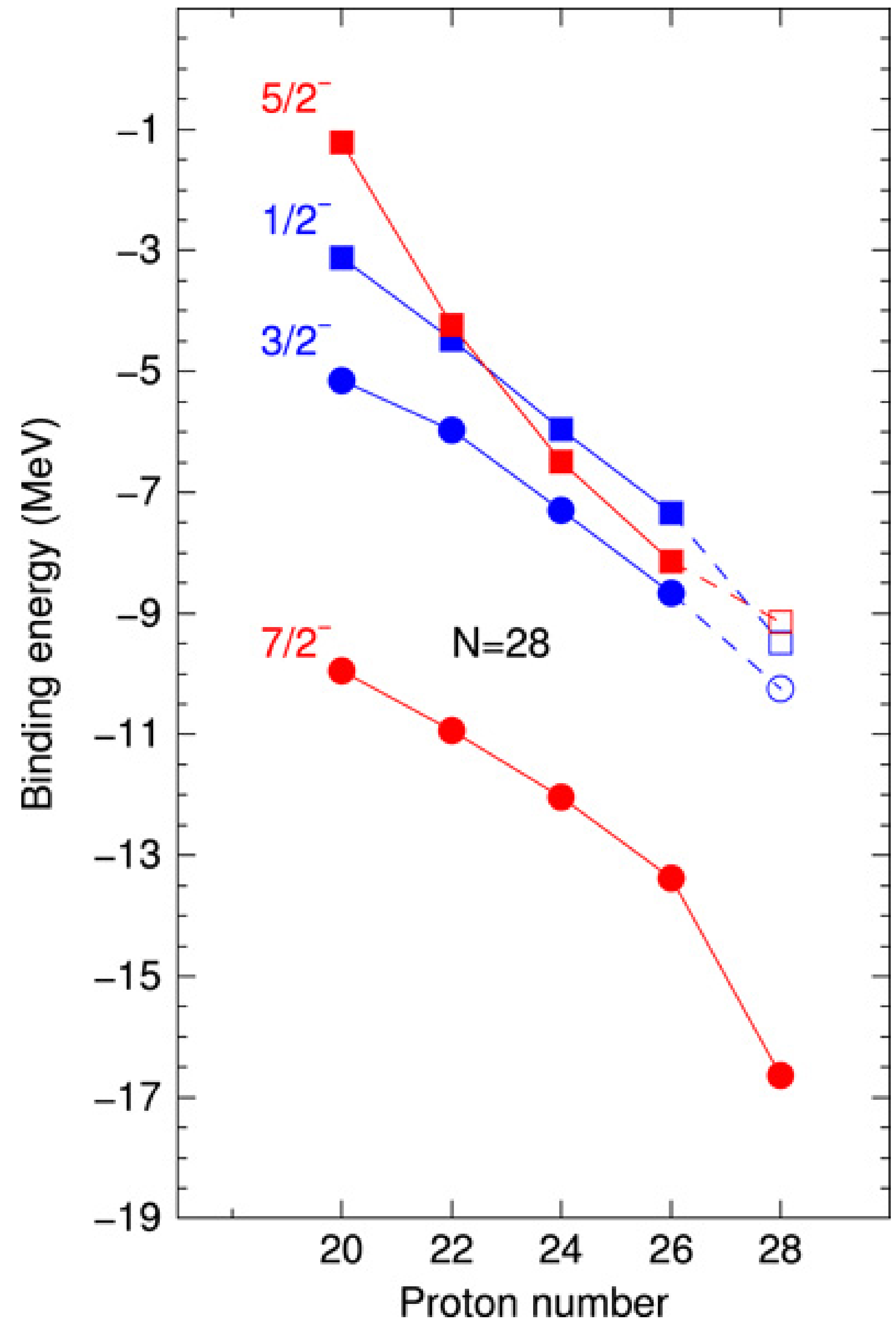
3) Why do hadrons have spin symmetry whereas nuclei have pseudo-spin symmetry?

HAPPY BIRTHDAY JAMES

And lollipops . . .



$N = 20$ isotones with $8 < Z < 20$



N = 28 isotones

Experimental Data

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Isovector Effects

σ, ω isoscalar mesons

ρ isovector meson

$$V_V^{\pi} = V_{\omega} - (N - Z) V_{\rho} \, ,$$

$$V_V^{\nu} = V_{\omega} + (N - Z) V_{\rho}$$

As neutron excess increases

$V_V^{\pi} + V_S^{\pi}$ **increases**

$V_V^{\nu} + V_S^{\nu}$ **decreases**

Pseudospin symmetry may improve for neutrons for neutron rich nuclei measured in RIA experiments!

Energy Eigenvalues in the Spin and Spherical Symmetry Limit

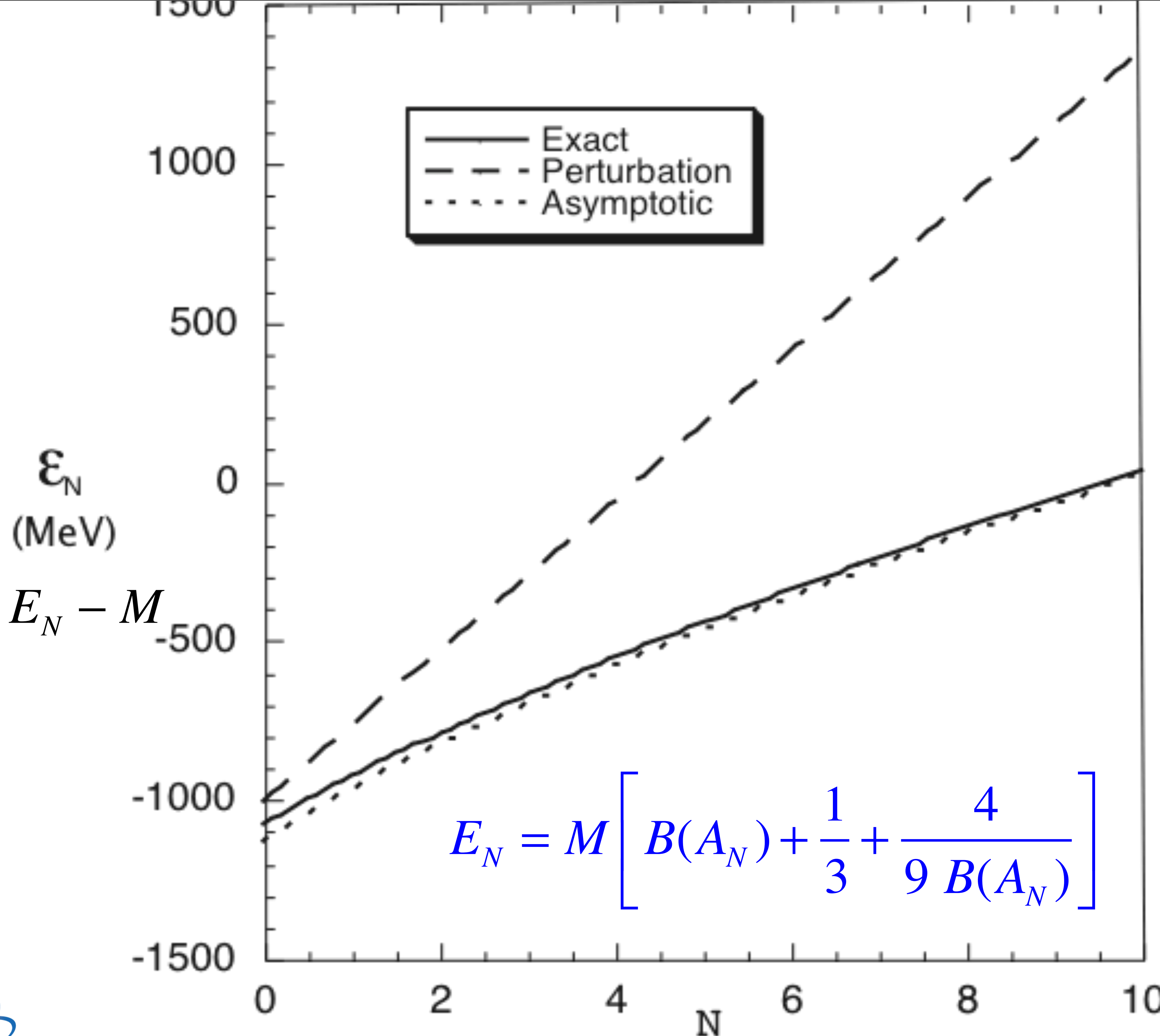
For the mass large compared to the potential, the eigenenergies go approximately like

$$E_N \approx M\,\left(1 + \frac{C\,(N + \frac{3}{2})}{\sqrt{2}} + \cdots\right)$$

that is linearly with N like the non-relativistic spectrum. For the mass small the spectrum goes like

$$E_N \approx M\,\left([C\,(N + \frac{3}{2})]^{\frac{2}{3}} + \frac{1}{3} + \cdots\right)$$

or approximately like N to the 2/3 power. Therefore the harmonic oscillator in the relativistic limit is not harmonic!



$$\varepsilon_N = E_N - M$$

$$E_N = M\left[B(A_N) + \frac{1}{3} + \frac{4}{9\,B(A_N)}\right]$$

Quadrupole Generators

Non-relativistic quadrupole generator:

$$q_m = \frac{1}{\hbar M \omega}\sqrt{\frac{3}{2}}\left(2M^2\omega^2[rr]_m^{(2)} + [pp]_m^{(2)}\right)$$

From the examples of the spin and orbital angular mometum the following ansatz seems plausible but, in fact, does not work:

$$Q_m = \begin{pmatrix} q_m & 0 \\ 0 & U_p\, q_m\, U_p \end{pmatrix}$$

Quadrupole Generators

$$Q_m = \begin{pmatrix} (Q_m)_{11} & (Q_m)_{12}\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p}\,(Q_m)_{21} & \vec{\sigma}\cdot\vec{p}\,(Q_m)_{22}\,\vec{\sigma}\cdot\vec{p} \end{pmatrix},$$

Conditions for Generators to Commute with the Hamiltonian

$$[Q_m, H] = 0$$

Conditions for Generators to Commute with the Hamiltonian

$$[Q_m, H] = 0$$

implies that

$$(Q_m)_{12} = (Q_m)_{21},$$

$$2[(Q_m)_{11}, V] + [(Q_m)_{12}, p^2] = 0,$$

$$2[(Q_m)_{12}, V] + [(Q_m)_{22}, p^2] = 0,$$

$$(Q_m)_{11} = (Q_m)_{12}\, 2(V + M) + (Q_m)_{22}\, p^2.$$

A solution is:

$$Q_m = \lambda_2 \begin{pmatrix} M\omega^2\,(M\omega^2\, r^2 + 2M)[rr]_m^{(2)} + [pp]_m^{(2)} & M\omega^2\,[rr]_m^{(2)}\,\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p}\,M\omega^2\,[rr]_m^{(2)} & [pp]_m^{(2)} \end{pmatrix}$$

Number of Quanta Operator

$$\hat{N} = \lambda_0 \begin{pmatrix} M\omega^2 (M\omega^2 r^2 + 2M)r^2 + p^2 & M\omega^2 r^2 \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} \, M\omega^2 r^2 & p^2 \end{pmatrix} - \frac{3}{2}$$

$$\lambda_0 = \frac{1}{2\hbar\sqrt{(H+M)M\omega^2}}$$

U(3) Commutation Relations

$$[\hat{N},\vec{L}] = [\hat{N},Q_m] = 0,$$

$$[\vec{L},\vec{L}]^{(t)} = -\sqrt{2}\,\vec{L}\,\delta_{t,1},$$

$$[\vec{L},Q]^{(t)} = -\sqrt{6}\,Q\,\delta_{t,2},$$

$$[Q,Q]^{(t)} = 3\sqrt{10}\,\vec{L}\,\delta_{t,1}.$$