POSITRONIUM IN BASIS LIGHT FRONT QUANTIZATION

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INTRODUCTION

- Light Front Bound State Equation

$$P^2\left|\Psi\right\rangle = M^2\left|\Psi\right\rangle \qquad H \equiv P^2$$

  - Fock Sector Decomposition of State Vector

$$\left|e^-e^+\right\rangle_{phys} = a\left|e^-e^+\right\rangle + b\left|e^-e^+\gamma\right\rangle + c\left|e^-e^+\gamma\gamma\right\rangle + d\left|\gamma\right\rangle + e\left|e^-e^+e^-e^+\right\rangle + \cdots$$

  - In general: Large, sparse matrix diagonalization problem

- Apply suitably adapted methods from *ab initio* nuclear structure

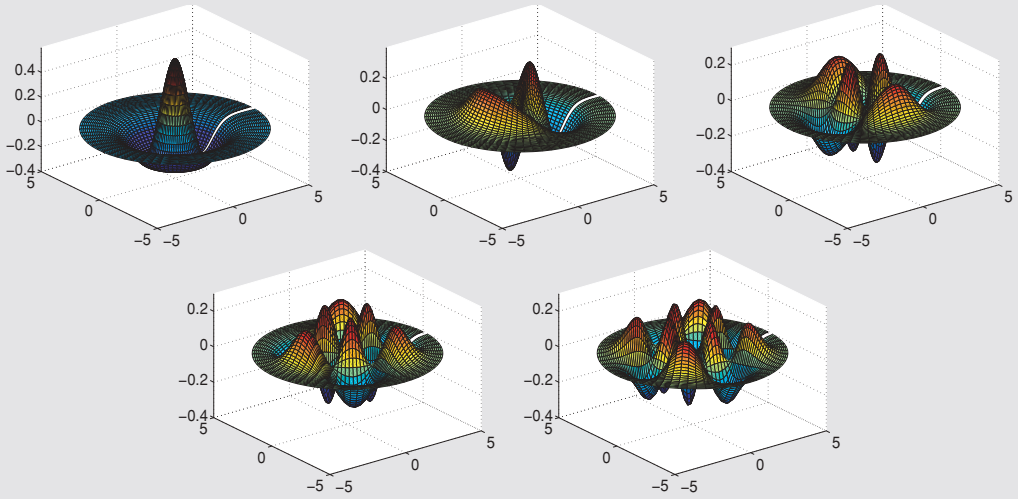
- Test Case: Positronium

BASIS LIGHT FRONT QUANTIZATION

- Transverse Harmonic Oscillator Basis
  - Eigenfunctions of 2D potential

$$V = \tfrac{1}{2} M\Omega^2 \mathbf{r}^2$$

  - Every value of "oscillator energy" Ω determines a complete basis
    - Energy scale $b \equiv \sqrt{M\Omega}$
  - Restrict number of oscillator quanta $\sum_i \left(2n_i + |m_i| + 1\right) \le N_{\max}$



BASIS LIGHT FRONT QUANTIZATION

- Discretize longitudinal momentum $p^+ = \frac{2\pi}{L} j$
  - Longitudinal box of length L
  - (anti)periodic boundary conditions

$$x_i = \frac{p_i^+}{P^+} = \frac{j_i}{K}$$

- Select value of total longitudinal momentum
  - Parameterize by dimensionless integer $P^+ = \frac{2\pi}{L} K$
  - K determines resolution of longitudinal momentum grid

- Pick total angular momentum projection – "M-scheme"

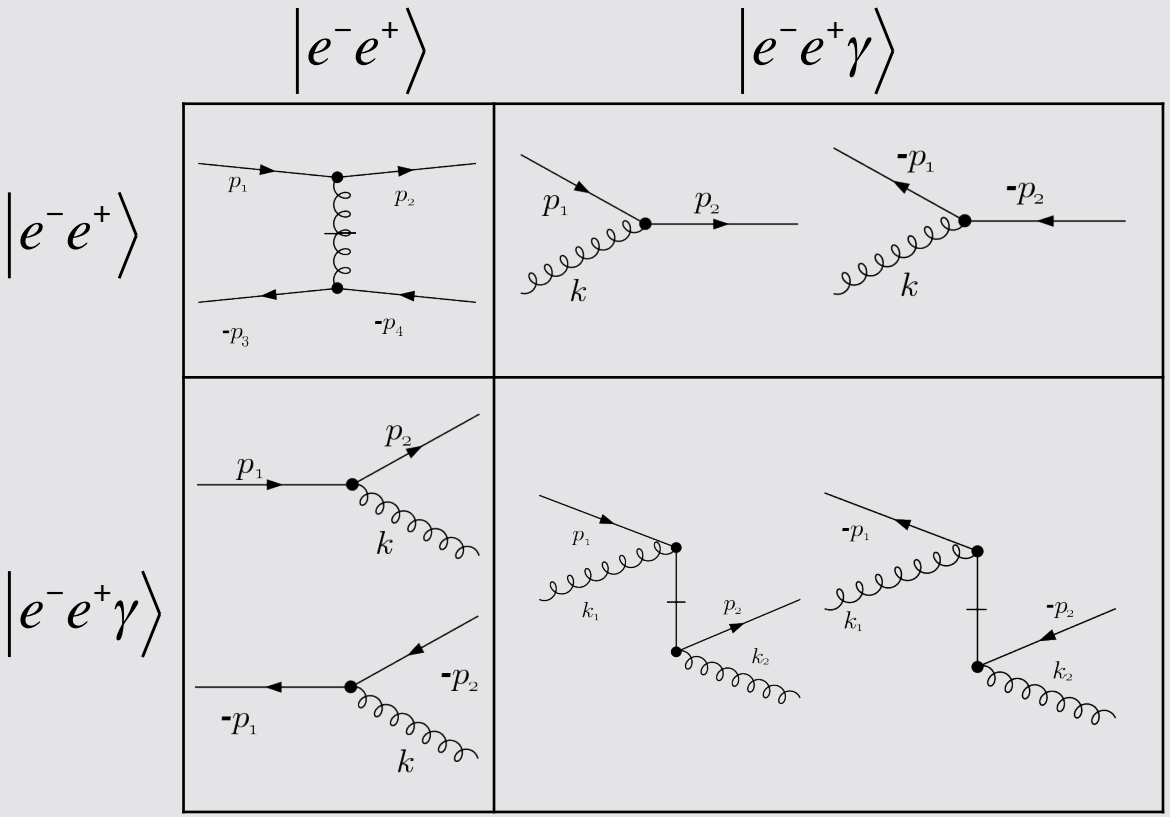
$$\sum_i \left(m_i + s_i \right) = M_J$$

- For QCD, choose color singlet configurations

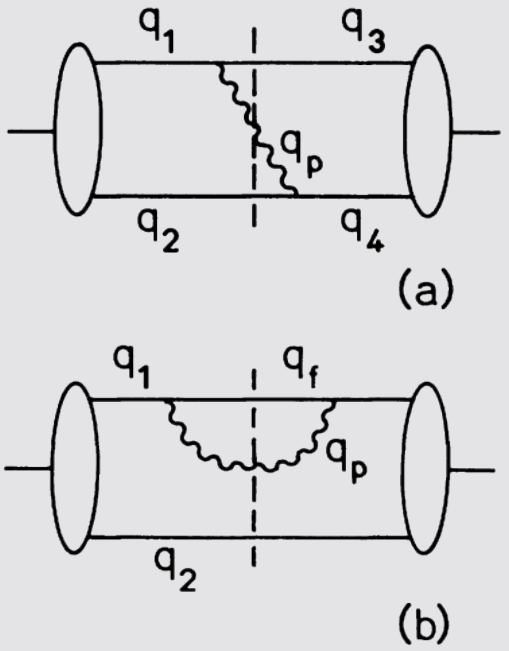
- Single-Particle Coordinates
  - Use specially chosen coordinates to guarantee exact CM factorization in truncated basis

$$\mathbf{q}_i = \frac{\mathbf{p}_i}{\sqrt{x_i}} \qquad \mathbf{s}_i = \sqrt{x_i}\mathbf{r}_i$$

HAMILTONIAN MATRIX



Iterated Interactions:



EFFECTIVE POTENTIAL

- Projection operators P and Q

$$P^2 \left|\Psi_i\right\rangle = M_i^2 \left|\Psi_i\right\rangle$$

$$H \equiv P^2$$

$$PHP \equiv H_{00}$$
$$PHQ \equiv H_{01}$$
$$QHP \equiv H_{10}$$
$$QHQ \equiv H_{11}$$

	$\left\|e^-e^+\right\rangle$	$\left\|e^-e^+\gamma\right\rangle$
$\left\|e^-e^+\right\rangle$	**PHP**	**PHQ**
$\left\|e^-e^+\gamma\right\rangle$	**QHP**	**QHQ**

- Eigenvalue equation can be rewritten:

$$H_{\text{eff}}(\omega)\left|\psi_i(\omega)\right\rangle_0 = \tilde{M}_i^2(\omega)\left|\psi_i(\omega)\right\rangle_0$$

$$H_{\text{eff}}(\omega) \equiv H_{00} + H_{01}\frac{1}{\omega - H_{11}}H_{10}$$

$$\omega = M_i^2$$

EFFECTIVE POTENTIAL

$$H_{\text{eff}}(\omega) \equiv H_{00} + H_{01} \frac{1}{\omega - H_{11}} H_{10}$$

H_{eff} = H_{00} + H_{01} $(\omega\text{-}H_{11})^{-1}$ H_{10}

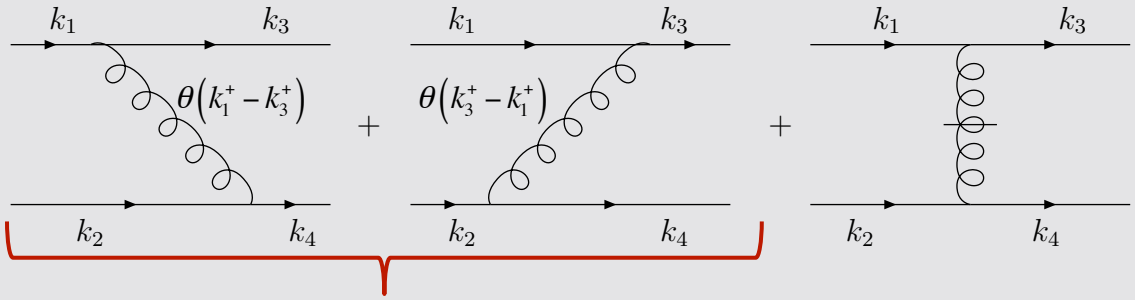
- In special case where H_{11} is diagonal

$$\langle f | H_{eff}(\omega) | i \rangle = \langle f | H_{00} | i \rangle + \sum_{|n\rangle} \frac{\langle f | H_{01} | n \rangle \langle n | H_{10} | i \rangle}{\omega - \langle n | H_{11} | n \rangle}$$

- Resolvent operator $(\omega\text{-}H_{11})^{-1}$ not diagonal in H.O. basis
  - Work in momentum space initially

SMALL-x DIVERGENCES

- Perturbative calculation of scattering amplitude



$$\propto \frac{1}{\left(x_1 - x_3\right)^2} \qquad \propto \frac{1}{\left(x_1 - x_3\right)^2}$$

- Each term has a divergence at small x
  - Divergences cancel in perturbative calculation
  - Result is identical to scattering amplitude obtained via usual Feynman rules in equal-time quantization

TWO-BODY EFFECTIVE INTERACTION

- Consider just second term of effective interaction

$$\left\langle f\left|H_{eff}^{(2)}(\omega)\right|i\right\rangle = \sum_{|n\rangle} \frac{\left\langle f\left|H_{01}\right|n\right\rangle\left\langle n\left|H_{10}\right|i\right\rangle}{\omega - E_n}$$

  - Sum reduces to sum over polarization states of photon

$$\sum_{\lambda_g} \varepsilon_\mu(k_g, \lambda_g)\, \varepsilon_\nu^*(k_g, \lambda_g) = -g_{\mu\nu} + (k_{g,\mu}\eta_\nu + k_{g,\nu}\eta_\mu)/k_g^\kappa \eta_\kappa$$

$$\eta^\mu = (\eta^+, \boldsymbol{\eta}_\perp, \eta^-) = (0, \mathbf{0}_\perp, 2)$$
$$\eta^2 \equiv \eta^\mu \eta_\mu = 0$$
$$k\eta = k^+$$

  - Expected cancellation is achieved only if:

$$\omega = \frac{E_i + E_f}{2}$$

EXPRESSION FOR EFFECTIVE POTENTIAL

- Final result

$$\left\langle f\left|H_{eff}^{(2)}\left(\omega=\tfrac{E_i+E_f}{2}\right)\right|i\right\rangle=\alpha\frac{\delta_{x_1+x_2}^{x_1'+x_2'}}{K}\int\frac{\mathrm{d}^2q_1}{(2\pi)^2}\frac{\mathrm{d}^2q_2}{(2\pi)^2}\frac{\mathrm{d}^2q_1'}{(2\pi)^2}\frac{\mathrm{d}^2q_2'}{(2\pi)^2}(2\pi)^2\,\delta^{(2)}\left(\sqrt{x_1}q_1+\sqrt{x_2}q_2-\sqrt{x_1'}q_1'+\sqrt{x_2'}q_2'\right)$$
$$\cdot\frac{\Psi_{n_1}^{m_1}(q_1)\Psi_{n_2}^{m_2}(q_2)\Psi_{n_1'}^{*m_1'}(q_1')\Psi_{n_2'}^{*m_2'}(q_2')\bar{u}(1')\gamma^\mu u(1)\bar{v}(2)\gamma_\mu v(2')}{\frac{(x_1-x_1')}{2}\left[\left(\frac{x_1q_1^2+m^2}{x_1}+\frac{x_1'q_1'^2+m^2}{x_1'}-\frac{\left(\sqrt{x_1}q_1-\sqrt{x_1'}q_1'\right)^2+\mu^2}{x_1-x_1'}\right)-\left(\frac{x_2q_2^2+m^2}{x_2}+\frac{x_2'q_2'^2+m^2}{x_2'}-\frac{\left(\sqrt{x_2}q_2-\sqrt{x_2'}q_2'\right)^2+\mu^2}{x_2-x_2'}\right)\right]}$$

$$I=-\frac{1}{x_1+x_2}\cdot\frac{1}{2\pi}\cdot\delta_{m_1+m_2}^{m_1'+m_2'}\cdot\sum(TM)(TM)(TM)\int\mathrm{d}x\mathrm{d}y\frac{e^{-(x+y)/2}L(x)L(y)}{(\alpha-\beta)^2x+(\alpha+\beta)^2y+\frac{\Delta}{b^2}+\varepsilon}$$

$$\alpha\equiv\sqrt{\frac{x_1'x_2}{x_1+x_2}}$$

$$\beta\equiv\sqrt{\frac{x_2'x_1}{x_1+x_2}}$$

$$\Delta\equiv m^2(x_1-x_1')^2\left[\frac{1}{x_1'x_1}+\frac{1}{x_2'x_2}\right]\geq 0$$

$$\varepsilon\equiv 2\frac{\mu^2}{b^2}$$

- 3-fold summation
- 2D Numerical Integral
- Infrared Cutoff

BLFQ RESULTS FOR POSITRONIUM

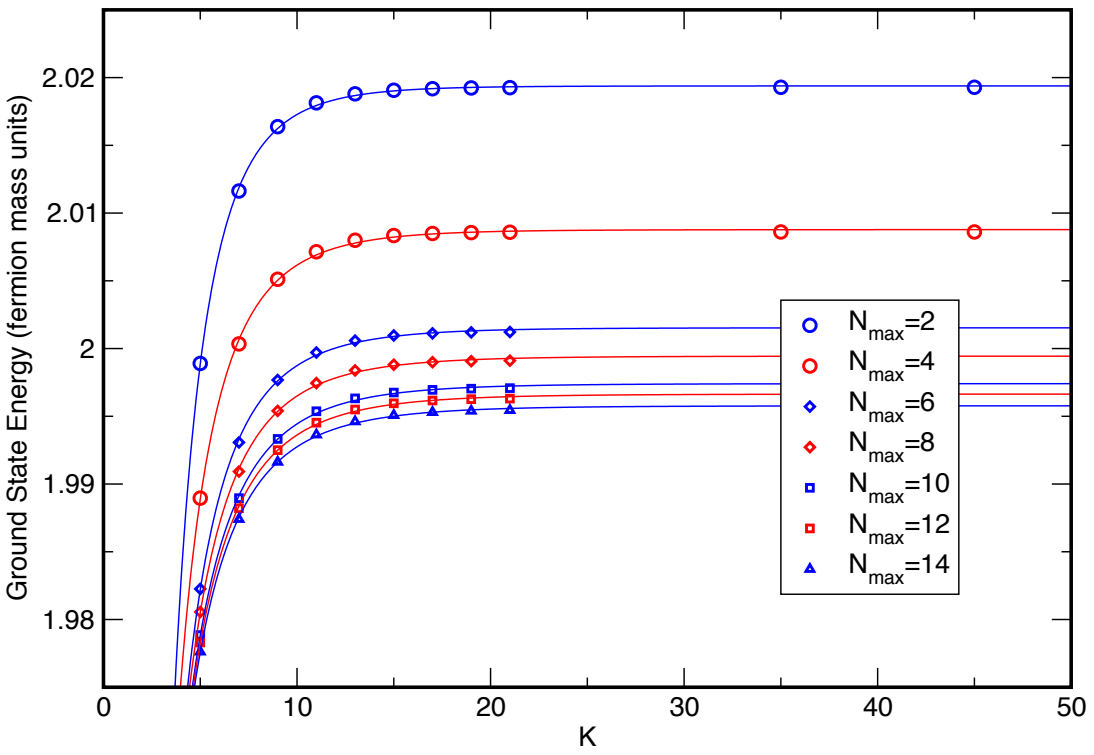
$\alpha = 0.3$

$\mu = 0.1 m_f$

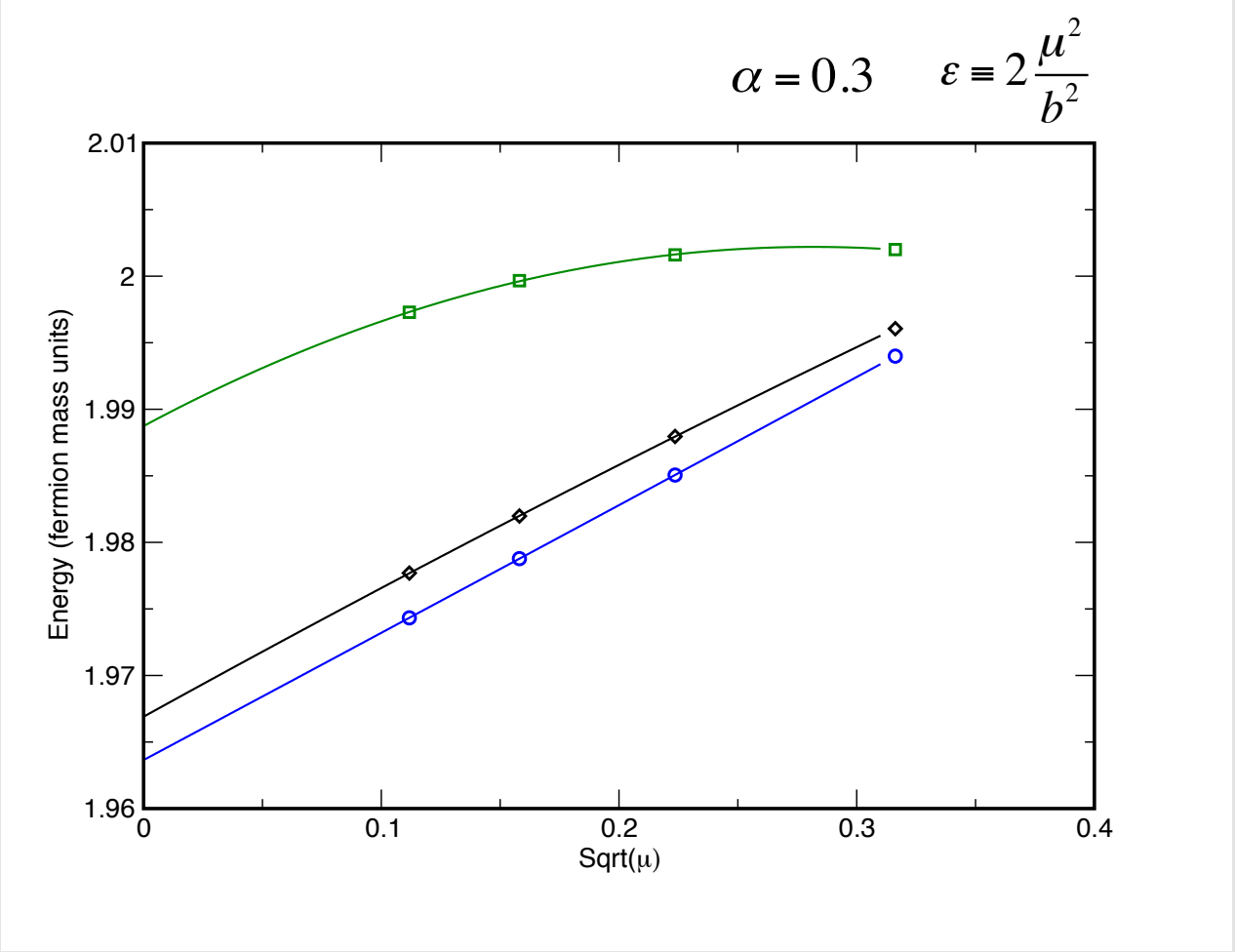
$b = 0.5 m_f$

$$\varepsilon \equiv 2\frac{\mu^2}{b^2} = 0.08$$

$$E = a + be^{-c\sqrt{K}}$$



ε → 0 LIMIT


$\alpha = 0.3$
$\varepsilon \equiv 2\frac{\mu^2}{b^2}$
2.01
2
1.99
1.98
1.97
1.96
Energy (fermion mass units)
0
0.1
0.2
0.3
0.4
Sqrt(μ)

COMPARISON

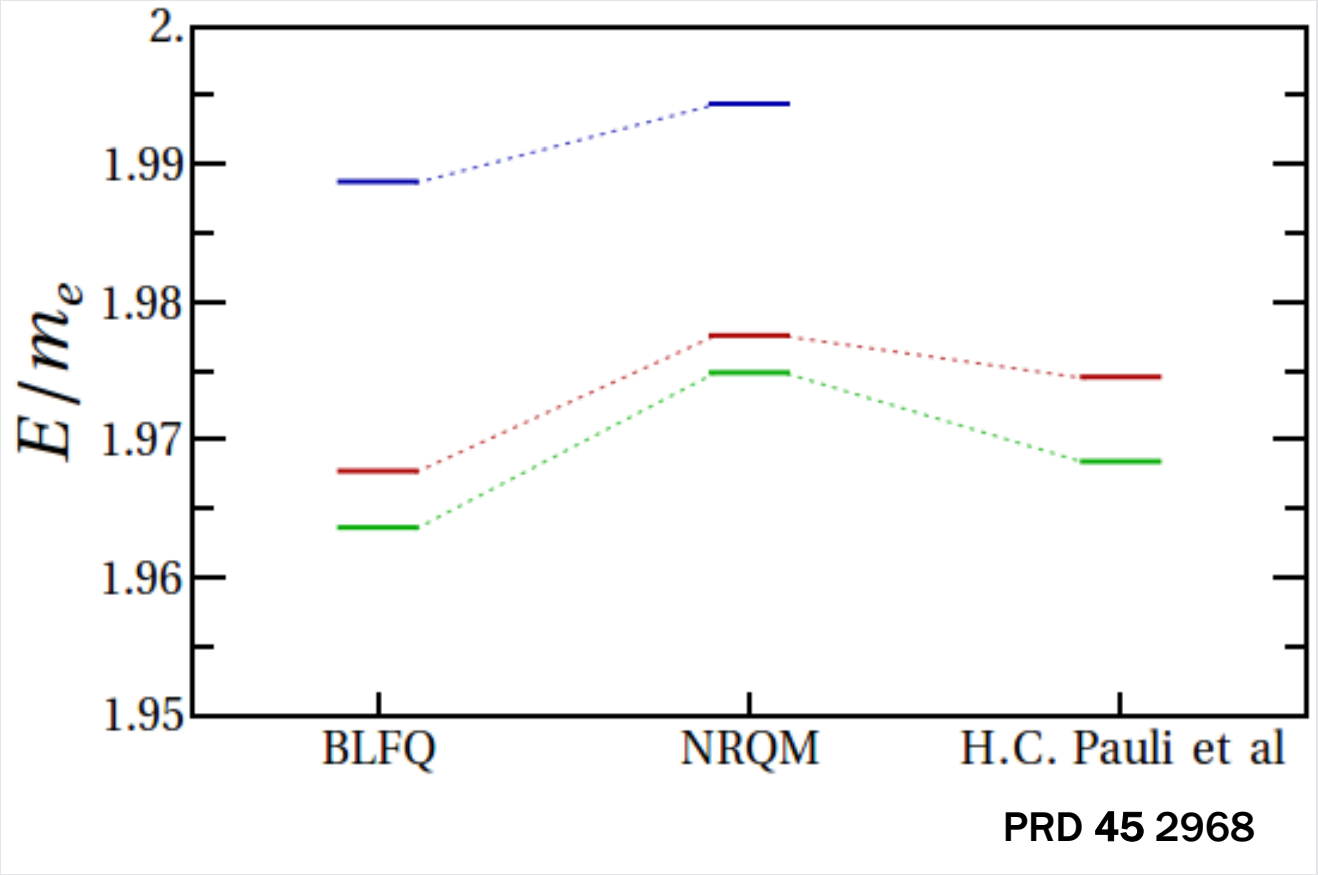
$\alpha = 0.3$

Hyperfine Splitting

$$M_t = 2 - \frac{\alpha^2}{4}\left(1 - \frac{1}{48}\alpha^2\right)$$

$$M_s = 2 - \frac{\alpha^2}{4}\left(1 + \frac{63}{48}\alpha^2\right)$$

$$\varepsilon \equiv 2\frac{\mu^2}{b^2}$$



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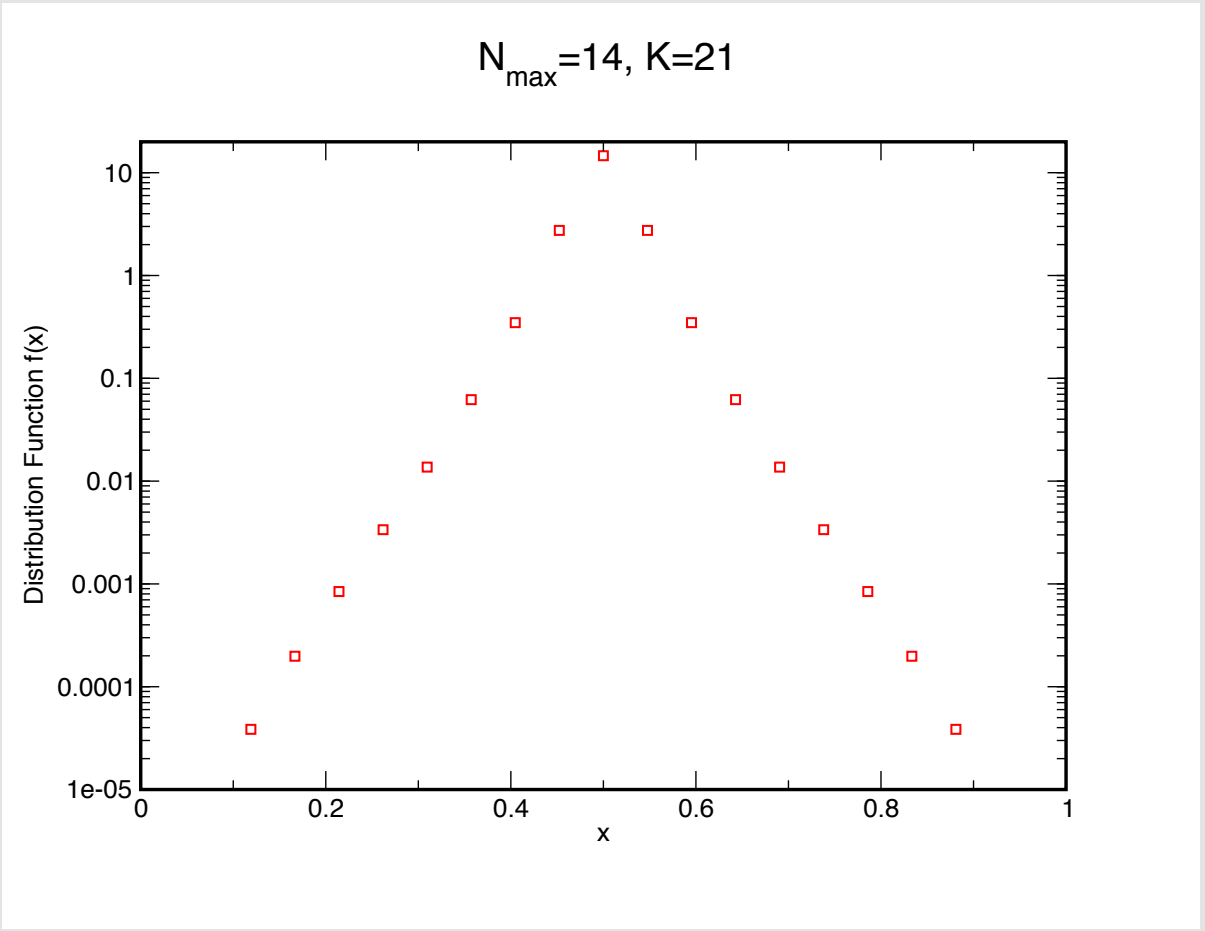
GROUND STATE DISTRIBUTION FUNCTION

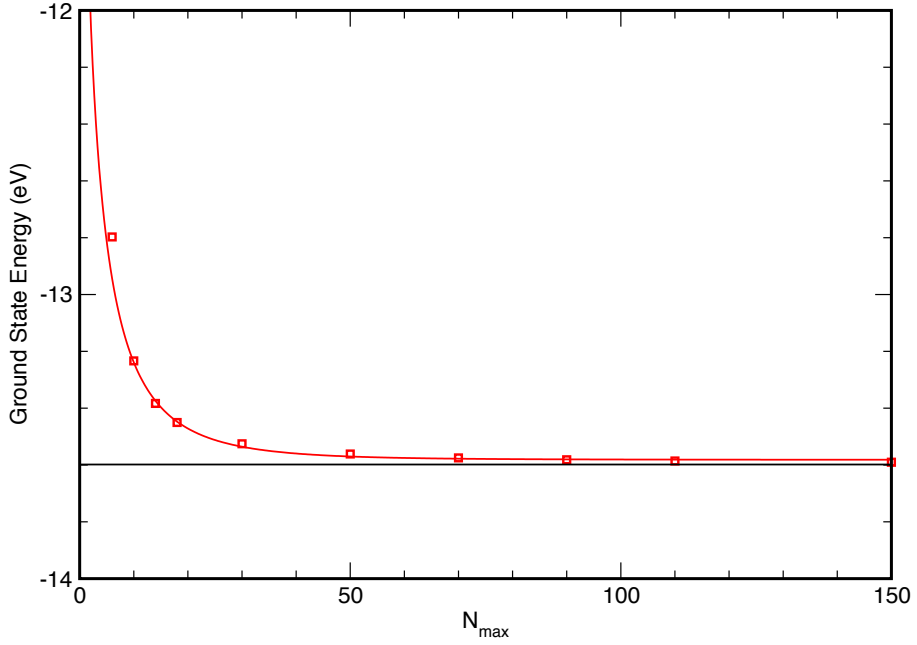


SUMMARY

- Effective two-body photon exchange interaction
  - Cancellation of Light Front small-x divergences obtained only for a particular value of arbitrary parameter ω
- Finite result in continuum limit
  - Results for binding energy and hyperfine splitting of ground state are the same order of magnitude as non-relativistic QM predictions
  - Overbinding for large α also found by H.C. Pauli's group (DLCQ)
- Future Work:
  - Heavy Quarkonia
  - Explicit dynamical photons in basis
    - Non-perturbative renormalization

HARMONIC OSCILLATOR BASIS

- Harmonic Oscillator Basis for Coulomb Problem
  - Wrong asymptotic behavior $e^{-\alpha r^2}$ vs. $e^{-\beta r}$



$$E = a + be^{-c\sqrt{N_{\max}}}$$

VARIATIONAL CALCULATION

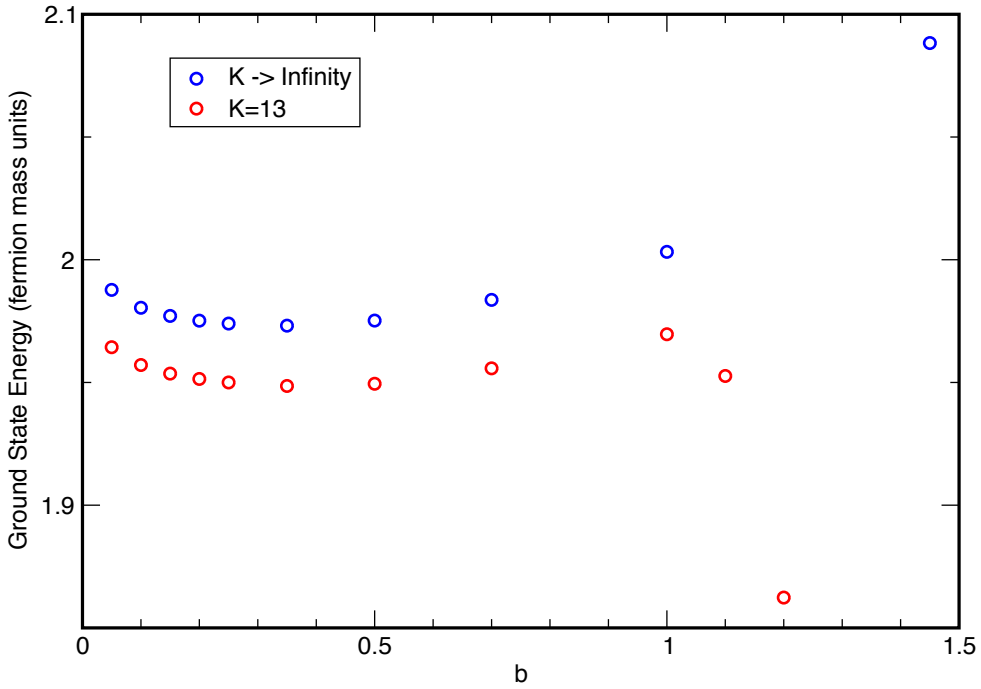
- $\varepsilon \longrightarrow 0$ limit is safe provided $K \longrightarrow \infty$

$$\alpha = 0.3$$

$$\mu = 0.005 \mathrm{m}_f$$

$$N_{\max} = 6$$

$$\varepsilon \equiv 2\frac{\mu^2}{b^2}$$



EXTRA

- Neglect instantaneous interactions when corresponding dynamical exchange is not present in model space