# LFQ of Large $N$ Scalar $\text{QCD}_2$ with a Higgs Potential

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**Abstract**

Recently Grinstein, Jora and Polosa have studied a model of large $N$ scalar quantum chromodynamics in one-space one-time dimension. This model admits a Bethe–Salpeter equation describing the discrete spectrum of $q\bar{q}$ bound states. They consider the gauge fields in the adjoint representation of SU($N$) and the scalar fields in the fundamental representation. The model is asymptotically free and linearly confining. The model provides a good framework for the description of a large class of tetraquark (diquark-antidiquark) states. Recently we have studied the light-front quantization of this model *without* a Higgs potential. In the present work, we study the light-front Hamiltonian, path integral and BRST formulations of this model in the presence of a Higgs potential.

**Keywords:** *Quantum chromodynamics (QCD); tetraquark states; diquark-antidiquark states; light-front quantization; Hamiltonian quantization; path integral quantization; BRST quantization*

## 1 Introduction

Study of multiquark states has been a subject of wide interest for a rather long time [1–15]. Their interpretation remains a challenging task and a number of phenomenological models [1–15] have been proposed to understand various experimental observations. Various possibilities of understanding the hadron structure different from the usual mesons and baryons [3,4] have been considered in the literature rather widely [1–15]. Some of these states find a rather more natural interpretation in terms of four quark states or tetraquark states [3–15]. By now it is widely perceived that not only the heavier states like the $X$, $Y$, $Z$ states have an exotic structure which find more natural explanation as tetraquark states or diquark-antidiquark ($Q\bar{Q}$) states [3–15], but even the light scalar mesons are also most likely the lightest particles with an exotic structure also to be understood as $Q\bar{Q}$ or tetraquark states (because they cannot be classified as standard $q\bar{q}$ mesons) [1–14].

Very recently 't Hooft, Isidori, Maini, Polosa and Riquer [13] and others [2–5], have shown how one could explain the decays of the light scalar mesons by assuming a dominant $Q\bar{Q}$ structure for the lightest scalar mesons, where the diquark ($Q$) is being taken to be a spin zero antitriplet color state [1–5]. Further, Grinstein,

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Jora and Polosa [14] have studied a model of large $N$ scalar quantum chromodynamics (QCD) [1–15] in one-space one-time dimensions. Their model admits [14] a Bethe–Salpeter equation describing the discrete spectrum of $q\bar{q}$ bound states [1–5]. It is important to emphasize here that in the first approximation, the nonet formed by $f_0(980)$, $a_0(980)$, $\kappa(900)$, $\sigma(500)$ is interpreted as the lowest $Q\bar{Q}$ multiplet [14], and the decuplet of scalar mesons with masses above 1 GeV, formed by $f_0(1370)$, $f_0(1500)$, $f_0(1710)$, $a_0(1450)$, $K_0(1430)$ and possibly containing the lowest glueball, is interpreted as the lowest $q\bar{q}$ scalar multiplet [12–14]. The work of Grinstein *et al.* [14] is seen to further support this hypothesis. In the work of Grinstein *et al.* [14], the gauge fields have been considered in the adjoint representation of SU($N$) and the scalar fields in the fundamental representation. The theory is asymptotically free and linearly confining and different aspects of this theory have been studied by several authors in various contexts [14].

In a recent paper we have studied [15] the light-front (LF) quantization (LFQ) of this theory [with a mass term for the complex scalar (diquark) field but without the Higgs potential], under appropriate LC gauge-fixing conditions. In the present work, we study the LF Hamiltonian [16], path integral [17–19] and BRST [20–22] formulations of this theory [14] in the presence of a Higgs potential on the LF (i. e., on the hyperplanes defined by the equal light-cone (LC) time $\tau = x^+ = (x^0 + x^1)/\sqrt{2} = constant$ [23–27]. The LF theory is seen to be gauge-invariant (GI) possessing a set of first-class constraints.

In our earlier work involving the LFQ of this theory [15], the theory was considered with a mass term for the complex scalar (diquark) field *but without a Higgs potential*, whereas we now study this theory *in the presence of the Higgs potential*. The motivation for doing this is to study the aspects related to the spontaneous symmetry breaking in the theory. Also, because the theory is GI, we also study its BRST quantization under appropriate BRST LC gauge-fixing. Actually, in the Hamiltonian and path integral quantization of a theory under some gauge-fixing conditions the gauge-invariance of the theory necessarily gets broken because the procedure of gauge-fixing converts the set of first-class constraints of the theory into a set of second-class ones. In view of this, in order to achieve the quantization of a GI theory, such that the gauge-invariance of the theory is maintained even under gauge-fixing, one of the possible ways is go to a more generalized procedure called the BRST quantization, where the extended gauge symmetry called the BRST symmetry is maintained even under gauge-fixing.

## 2 Some basics of the theory

In this section we consider the instant-form (IF) quantization (IFQ) of this model of large $N$ scalar QCD in the presence of a Higgs potential, studied by Grinstein, Jora and Polosa [without a Higgs potential but with a mass term for the complex scalar (diquark) field $\phi$] [14]. We absorb the mass term for the complex scalar (diquark) field $\phi$ in the definition of our Higgs potential. The bosonized action of the theory that we study is defined (suppressing the color indices) by the action:

$$S = \int \mathcal{L}(\phi, \phi^\dagger, A^\mu)\, d^2x, \tag{1a}$$

$$\mathcal{L} = \Big[ -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \partial_\mu \phi^\dagger \partial^\mu \phi + \big[ i\rho(\phi A^\mu \partial_\mu \phi^\dagger - \phi^\dagger A_\mu \partial^\mu \phi) + \rho^2 \phi^\dagger \phi A_\mu A^\mu \big] - V(|\phi|^2) \Big], \tag{1b}$$

$$V(|\phi|^2) = V(\phi^\dagger\phi) = \left[\mu^2(\phi^\dagger\phi) + \frac{\lambda}{6}(\phi^\dagger\phi)^2\right], \quad |\phi|^2 = (\phi^\dagger\phi), \quad \phi_0 \neq 0, \tag{1c}$$

$$F^{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu), \quad \rho = \frac{g}{\sqrt{N}}, \quad (-\mu^2 > 0\ ,\ \lambda > 0), \tag{1d}$$

$$g^{\mu\nu} = g_{\mu\nu} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mu,\ \nu = 0,\ 1 \qquad \text{(IFQ)}, \tag{1e}$$

$$g^{\mu\nu} = g_{\mu\nu} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mu,\ \nu = +,\ - \qquad \text{(LFQ)}. \tag{1f}$$

In the above Lagrangian density of the theory, the first term represents the kinetic energy of the gluon field, the second term represents the kinetic energy term for the scalar absorbed (diquark) field, the third term represents the interaction term for the scalar (diquark) field with the gluon field (the color indices have again been suppressed) and the last term represents the Higgs potential which is kept rather general, without making any specific choice for the parameters $\mu^2$ and $\lambda$. However, they are chosen such that the potential remains a double well potential with the vacuum expectation value $\phi_0 = \langle 0|\phi(x)|0\rangle \neq 0$, so as to allow the spontaneous symmetry breaking in the theory. Also, the mass term for the scalar (diquark) field has been absorbed in the definition of the Higgs potential. The values $\mu^2 = m^2$ and $\lambda = 0$ reproduce the theory of Grinstein, Jora and Polosa [14]. For obtaining the gauged theory under our present investigation, we have used the gauging prescription: $\partial_\mu \to D_\mu = (\partial_\mu + i\rho A_\mu)$ (where the color indices are being suppressed) (it is to be noted here that our work of Ref. [15] uses a different gauging prescription).

Also, in order to remain consistent with the work of Grinstein, Jora and Polosa [14], we have ignored the gluon self coupling term in our considerations (just like the work of Ref. [25]).

# 3 Instant-form quantization

In the instant-form quantization of the theory (with the metric tensor $g_{\mu\nu} = g^{\mu\nu} = diag(+1, -1)$; $\mu, \nu =\ 0, 1$), the theory is seen to possess a set of three constraints:

$$\begin{aligned} &\psi_1 = \Pi^0 \approx 0, \qquad \psi_2 = \left[\partial_1 E + i\rho(\phi\pi - \phi^\dagger\pi^\dagger)\right] \approx 0, \\ &\psi_3 = \left[2\rho^2 A_0 \pi^\dagger\phi^\dagger + i\rho A_1(\phi\partial_1\phi^\dagger + \phi^\dagger\partial_1\phi)\right] \approx 0, \end{aligned} \tag{2}$$

where the constraint $\psi_1$ is a primary constraint and the constraints $\psi_2$ and $\psi_3$ are the secondary Gauss-law constraints. Also, here $\pi$, $\pi^\dagger$, $\Pi^0$ and $E = \Pi^1$ are the momenta canonically conjugate respectively to $\phi$, $\phi^\dagger$, $A_0$ and $A_1$ (here, $A_0 \equiv A_0^a\tau^a$, $A_1 \equiv A_1^a\tau^a$, $\Pi^0 \equiv \Pi^{0a}\tau^a$, $E \equiv E^a\tau^a$). The symbol $\approx$ here denotes a weak equality in the sense of Dirac [16]. Further, these constraints are easily seen to form a set of second-class constraints because the matrix of the Poisson brackets among these constraints is a non-singular matrix implying that the theory is gauge-non-invariant. The canonical Hamiltonian density of this theory is:

$$\begin{aligned} \mathcal{H}_c = \bigg[&\frac{1}{2}(E)^2 - A_0\,\partial_1 E + \pi^\dagger\pi + \partial_1\phi^\dagger\partial_1\phi + \rho^2 A_1^2\phi^\dagger\phi \\ &- i\rho A_0(\phi\pi - \phi^\dagger\pi^\dagger) - i\rho A_1(\phi^\dagger\partial_1\phi - \phi\,\partial_1\phi^\dagger) + \mu^2(\phi^\dagger\phi) + \frac{\lambda}{6}(\phi^\dagger\phi)^2\bigg]. \end{aligned} \tag{3}$$

After including the primary constraint $\psi_1$ in the canonical Hamiltonian density with the help of the Lagrange multiplier field $u$, the total Hamiltonian density becomes:

$$\begin{aligned} \mathcal{H}_T = \bigg[&\Pi^0 u + \frac{1}{2}(E)^2 - A_0\,\partial_1 E + \pi^\dagger\pi + \partial_1\phi^\dagger\partial_1\phi + \rho^2 A_1^2\phi^\dagger\phi \\ &- i\rho A_0(\phi\pi - \phi^\dagger\pi^\dagger) - i\rho A_1(\phi^\dagger\partial_1\phi - \phi\,\partial_1\phi^\dagger) + \mu^2(\phi^\dagger\phi) + \frac{\lambda}{6}(\phi^\dagger\phi)^2\bigg]. \end{aligned} \tag{4}$$

The Hamilton's equations of motion of the theory that preserve the constraints of the theory in the course of time could be obtained from the total Hamiltonian density and are omitted here for the sake of brevity. The matrix $R_{\alpha\beta}$ of the Poisson brackets among the set of these constraints $\psi_i$ with $(i = 1, 2, 3)$ is seen to be singular, implying that the set of these constraints $\psi_i$ is first-class and that the theory under consideration is gauge-invariant. Consequently the theory is seen to possess the local vector gauge symmetry defined by the local vector gauge transformations:

$$\delta\phi = i\rho\beta\phi, \quad \delta\phi^\dagger = -i\rho\beta\phi^\dagger, \quad \delta A_0 = \partial_0\beta, \quad \delta A_1 = \partial_1\beta, \tag{5}$$

where $\beta \equiv \beta(x_0, x_1)$ is an arbitrary function of its arguments. This theory could now be quantized under some appropriate gauge-fixing conditions, e. g., under the time-axial or temporal gauge: $A_0 \approx 0$. The details of this IFQ are however, outside the scope of the present work [what actually happens is that one of the matrix elements of the matrix $R_{\alpha\beta}$ involves a linear combination of a Dirac distribution function $\delta(x^1-y^1)$ and its first derivative and finding its inverse is a rather non-trivial task]. We now proceed with the LFQ of this theory in the next section.

# 4 Light-front Hamiltonian and path integral quantization

For the LFQ, the bosonized action of the theory (suppressing the color indices) in LF coordinates $x^\pm := (x^0 \pm x^1)/\sqrt{2}$ reads:

$$S = \int \mathcal{L}\, dx^+ dx^-, \tag{6a}$$

$$\begin{aligned}\mathcal{L} = \Big[ &\frac{1}{2}(\partial_+ A^+ - \partial_- A^-)^2 + (\partial_+\phi^\dagger\partial_-\phi + \partial_-\phi^\dagger\partial_+\phi) - \mu^2(\phi^\dagger\phi) - \frac{\lambda}{6}(\phi^\dagger\phi)^2 \\ &+ i\rho A^+(\phi\,\partial_+\phi^\dagger - \phi^\dagger\partial_+\phi) + i\rho A^-(\phi\,\partial_-\phi^\dagger - \phi^\dagger\partial_-\phi) + 2\rho^2\phi^\dagger\phi\, A^+A^- \Big].\end{aligned} \tag{6b}$$

In the work of Ref. [14], the authors have studied the above action, after implementing the gauge-fixing condition (GFC) $A^+ \approx 0$ "strongly" in the above action. In contrast to this, we propose to study the theory defined by the above action, following the standard Dirac quantization procedure [16] and we do not fix any gauge at this stage. We instead consider this GFC ($A^+ \approx 0$) as one of the gauge constraints [16] which becomes strongly equal to zero only on the reduced hyper surface of the constraints and remains non-zero in the rest of the phase space of the theory and we do not set it strongly equal to zero in above equation.

We like to emphasize here that one of the salient features of Dirac quantization procedure [16] is that in this quantization the GFC's should be treated on par with other gauge-constraints of the theory which are only weakly equal to zero in the sense of Dirac [16], and they become strongly equal to zero only on the reduced hyper surface of the constraints of the theory and not in the rest of the phase space of the classical theory (in the corresponding quantum theory these weak equalities become the weak operator equalities).

Another thing to be noted here is that we have introduced the Higgs potential in our present work and we have absorbed the mass term for the scalar (diquark) field in the definition of our Higgs potential. This LF theory is seen to possess a set of four constraints:

$$\begin{aligned}&\chi_1 = \Pi^+ \approx 0, \quad \chi_2 = [\pi - \partial_-\phi^\dagger + i\rho A^+\phi^\dagger] \approx 0, \quad \chi_3 = [\pi^\dagger - \partial_-\phi - i\rho A^+\phi] \approx 0, \\ &\chi_4 = [\partial_-\Pi^- + i\rho(\phi\,\partial_-\phi^\dagger - \phi^\dagger\partial_-\phi) + 2\rho^2\phi^\dagger\phi\, A^+] \approx 0,\end{aligned} \tag{7}$$

where the constraints $\chi_1$, $\chi_2$ and $\chi_3$ are primary constraints and the constraint $\chi_4$ is the secondary Gauss-law constraint, which is obtained by demanding that the primary constraint $\chi_1$ be preserved in the course of time. The preservation of $\chi_2$, $\chi_3$ and $\chi_4$, for all times does not give rise to any further constraints. The theory is thus seen to possess only four constraints $\chi_i$ (with $i = 1, 2, 3, 4$). Also, here $\pi$, $\pi^\dagger$, $\Pi^+$ and $\Pi^-$ are the momenta canonically conjugate respectively to $\phi$, $\phi^\dagger$, $A^-$ and $A^+$ (here, $A^+ \equiv A^{+a}\tau^a$, $A^- \equiv A^{-a}\tau^a$, $\Pi_0 \equiv \Pi_{0a}\tau^a$, $E \equiv E^a\tau^a$). Now, the constraints $\chi_2$, $\chi_3$ and $\chi_4$ could however, be combined into a single constraint:

$$\eta = \left[\partial_-\Pi^- + i\rho(\phi\,\pi - \phi^\dagger\pi^\dagger)\right] \approx 0, \tag{8}$$

and with this modification, the new set of constraints of the theory could be written as:

$$\Omega_1 = \chi_1 = \Pi^+ \approx 0, \qquad \Omega_2 = \eta = \left[\partial_-\Pi^- + i\rho(\phi\pi - \phi^\dagger\pi^\dagger)\right] \approx 0. \tag{9}$$

Further, the matrix of the Poisson brackets among the constraints $\Omega_i$, with $i = 1, 2$ is seen to be a singular matrix implying that the set of constraints $\Omega_i$ is first-class and that the theory under consideration is gauge-invariant. The canonical Hamiltonian density for this LF theory is:

$$\begin{aligned}\mathcal{H}_c = \bigg[\frac{1}{2}(\Pi^-)^2 + \Pi^-(\partial_-A^-) + \mu^2(\phi^\dagger\phi) + \frac{\lambda}{6}(\phi^\dagger\phi)^2 \\ - i\rho A^-(\phi\,\partial_-\phi^\dagger - \phi^\dagger\partial_-\phi) - 2\rho^2\phi^\dagger\phi\,A^+A^-\bigg].\end{aligned} \tag{10}$$

After including the primary constraints $\chi_1$, $\chi_2$ and $\chi_3$ in the canonical Hamiltonian density $\mathcal{H}_c$ with the help of the Lagrange multiplier fields $u$, $v$ and $w$, the total Hamiltonian density could be written as:

$$\begin{aligned}\mathcal{H}_T = \bigg[(\Pi^+)u + (\pi - \partial_-\phi^\dagger + i\rho A^+\phi^\dagger)v + (\pi^\dagger - \partial_-\phi - i\rho A^+\phi)w \\ + \mu^2(\phi^\dagger\phi) + \frac{\lambda}{6}(\phi^\dagger\phi)^2 + \frac{1}{2}(\Pi^-)^2 + \Pi^-\partial_-A^- \\ - i\rho A^-(\phi\,\partial_-\phi^\dagger - \phi^\dagger\partial_-\phi) - 2\rho^2\phi^\dagger\phi\,A^+A^-\bigg].\end{aligned} \tag{11}$$

The Hamilton's equations of motion of the theory that preserve the constraints of the theory in the course of time could be obtained from the total Hamiltonian density. Also, the divergence of the vector gauge current density of the theory is seen to vanish, implying that the theory possesses at the classical level a local vector-gauge symmetry. The action of the theory is indeed seen to be invariant under the local vector gauge transformations:

$$\begin{aligned}&\delta\phi = -i\rho\beta\phi, \quad \delta\phi^\dagger = i\rho\beta\phi^\dagger, \quad \delta A^- = \partial_+\beta, \quad \delta A^+ = \partial_-\beta, \\ &\delta\pi = \left[\rho^2\beta\phi^\dagger A^+ + i\rho\beta\partial_-\phi^\dagger\right], \quad \delta\pi^\dagger = \left[\rho^2\beta\phi A^+ - i\rho\beta\partial_-\phi\right], \\ &\delta u = \delta v = \delta w = \delta\Pi^+ = \delta\Pi^- = \delta\Pi_u = \delta\Pi_v = \delta\Pi_w = 0,\end{aligned} \tag{12}$$

where $\beta \equiv \beta(x^+, x^-)$ is an arbitrary function of its arguments and $\Pi_u$, $\Pi_v$ and $\Pi_w$ are the momenta canonically conjugate to the Lagrange multiplier fields $u$, $v$ and $w$ respectively, which are treated here as dynamical fields.

The theory could now be quantized, e. g., under the GFC's: $\zeta_1 = A^+ \approx 0$, $\zeta_2 = A^- \approx 0$, where the gauge $A^+ \approx 0$ represents the LC time-axial or temporal gauge and the gauge $A^- \approx 0$ represents the LC Coulomb gauge and both of these gauges are physically important gauges. The matrix $R_{\alpha\beta}$ of the Poisson brackets among the set of constraints $\Omega_i$ with $i = 1,\, 2$ is seen to be nonsingular with the determinant given by

$$\Big[\big|\,|\det(R_{\alpha\beta})|\,\big|\Big]^{\frac{1}{2}} = \Big[\big[\delta^{'}(x^- - y^-)\big]\,\big[\delta(x^- - y^-)\big]\Big]. \tag{13}$$

Finally, following the Dirac quantization procedure in the Hamiltonian formulation, the non-vanishing equal light-cone-time commutators of the theory, under the GFC's $A^+ \approx 0$ and $A^- \approx 0$ are obtained as:

$$\big[\phi(x^+,x^-),\pi(x^+,y^-)\big] = i\delta(x^- - y^-), \tag{14}$$

$$\big[\phi^\dagger(x^+,x^-),\pi^\dagger(x^+,y^-)\big] = i\delta(x^- - y^-), \tag{15}$$

$$\big[\phi(x^+,x^-),\Pi^-(x^+,y^-)\big] = \frac{1}{2}\rho\phi\epsilon(x^- - y^-), \tag{16}$$

$$\big[\phi^\dagger(x^+,x^-),\Pi^-(x^+,y^-)\big] = -\frac{1}{2}\rho\phi^\dagger\epsilon(x^- - y^-), \tag{17}$$

$$\big[\pi(x^+,x^-),\Pi^-(x^+,y^-)\big] = \frac{1}{2}\rho\pi\epsilon(x^- - y^-), \tag{18}$$

$$\big[\pi^\dagger(x^+,x^-),\Pi^-(x^+,y^-)\big] = -\frac{1}{2}\rho\pi^\dagger\epsilon(x^- - y^-), \tag{19}$$

$$\big[\Pi^-(x^+,x^-),\phi(x^+,y^-)\big] = \frac{1}{2}\rho\phi\epsilon(x^- - y^-), \tag{20}$$

$$\big[\Pi^-(x^+,x^-),\phi^\dagger(x^+,y^-)\big] = -\frac{1}{2}\rho\phi^\dagger\epsilon(x^- - y^-), \tag{21}$$

$$\big[\Pi^-(x^+,x^-),\pi(x^+,y^-)\big] = -\frac{1}{2}\rho\pi\epsilon(x^- - y^-), \tag{22}$$

$$\big[\Pi^-(x^+,x^-),\pi^\dagger(x^+,y^-)\big] = \frac{1}{2}\rho\pi^\dagger\epsilon(x^- - y^-). \tag{23}$$

The first-order Lagrangian density $\mathcal{L}_{I0}$ of the theory is:

$$\mathcal{L}_{I0} = \bigg[\frac{1}{2}(\Pi^-)^2 + \partial_+\phi^\dagger\partial_-\phi + \partial_-\phi^\dagger\partial_+\phi + 2\rho^2\phi^\dagger\phi A^+A^- - \mu^2\phi^\dagger\phi - i\rho A^-(\phi^\dagger\partial_-\phi - \phi\partial_-\phi^\dagger) - i\rho A^+(\phi^\dagger\partial_+\phi - \phi\,\partial_+\phi^\dagger) - \frac{\lambda}{6}(\phi^\dagger\phi)^2\bigg]. \tag{24}$$

In the path integral formulation [17–19], the transition to quantum theory is made by writing the vacuum to vacuum transition amplitude for the theory called the generating functional $Z[J_k]$. For the present theory, under the GFC's $\zeta_1 = A^+ \approx 0$ and $\zeta_2 = A^- \approx 0$ and in the presence of the external sources $J_k$, it reads:

$$Z[J_k] = \int [d\mu]\exp\bigg[i\int d^2x\,\Big(J_k\Phi^k + \pi\partial_+\phi + \pi^\dagger\partial_+\phi^\dagger + \Pi^+\partial_+A^- + \Pi^-\partial_+A^+ + \Pi_u\partial_+u + \Pi_v\partial_+v + \Pi_w\partial_+w - \mathcal{H}_T\Big)\bigg]. \tag{25}$$

Here, the phase space variables of the theory are $\Phi^k \equiv (\phi,\phi^\dagger,A^-,A^+,u,v,w)$ with the corresponding respective canonical conjugate momenta: $\Pi_k \equiv (\pi,\pi^\dagger,\Pi^+,\Pi^-,\Pi_u,$ $\Pi_v,\Pi_w)$. The functional measure $[d\mu]$ of the generating functional $Z[J_k]$ under the above gauge-fixing is obtained as:

$$[d\mu] = [\delta'(x^- - y^-)\,\delta(x^- - y^-)][d\phi][d\phi^\dagger][dA^+][dA^-][du][dv][dw] \times [d\pi][d\pi^\dagger][d\Pi^-][d\Pi^+][d\Pi_u][d\Pi_v][d\Pi_w]\,\delta[\Pi^+ \approx 0]\,\delta[A^- \approx 0] \times \delta\big[\big(\partial_-\Pi^- + i\rho(\phi\pi - \phi^\dagger\pi^\dagger)\big) \approx 0\big]\,\delta[A^+ \approx 0]. \tag{26}$$

# 5 Light-front BRST quantization

For the BRST formulation of the model, we rewrite the theory as a quantum system that possesses the generalized gauge invariance called BRST symmetry. For this, we first enlarge the Hilbert space of our gauge-invariant theory and replace the notion of gauge-transformation, which shifts operators by $c$-number functions, by a BRST transformation, which mixes operators with Bose and Fermi statistics. We then introduce new anti-commuting variable $c$ and $\bar{c}$ (Grassmann numbers on the classical level and operators in the quantized theory) and a commuting variable $b$ such that [20–22]:

$$\begin{aligned} &\hat{\delta}\phi = -i\rho c\phi, \quad \hat{\delta}\phi^\dagger = i\rho c\phi^\dagger, \quad \hat{\delta}A^- = \partial_+ c, \quad \hat{\delta}A^+ = \partial_- c, \\ &\hat{\delta}\pi = \left[\rho^2 c\,\phi^\dagger A^+ + i\rho\, c\,\partial_-\phi^\dagger\right], \quad \hat{\delta}\pi^\dagger = \left[\rho^2 c\,\phi A^+ - i\rho\, c\,\partial_-\phi\right], \\ &\hat{\delta}u = \hat{\delta}v = \hat{\delta}w = \hat{\delta}\Pi^+ = \hat{\delta}\Pi^- = \hat{\delta}\Pi_u = \hat{\delta}\Pi_v = \hat{\delta}\Pi_w = 0, \\ &\hat{\delta}c = 0, \quad \hat{\delta}\bar{c} = b, \quad \hat{\delta}b = 0, \end{aligned} \tag{27}$$

with the property $\hat{\delta}^2 = 0$. We now define a BRST-invariant function of the dynamical phase space variables of the theory to be a function $f$ such that $\hat{\delta}f = 0$. Now the BRST gauge-fixed quantum Lagrangian density $\mathcal{L}_{BRST}$ for the theory could be obtained by adding to the first-order Lagrangian density $\mathcal{L}_{I0}$, a trivial BRST-invariant function, e. g., as follows:

$$\begin{aligned} \mathcal{L}_{BRST} = \Bigg[&\frac{1}{2}(\Pi^-)^2 + \partial_+\phi^\dagger\partial_-\phi + \partial_-\phi^\dagger\partial_+\phi - i\rho A^-(\phi^\dagger\partial_-\phi - \phi\,\partial_-\phi^\dagger) \\ &- \frac{\lambda}{6}(\phi^\dagger\phi)^2 - \mu^2\phi^\dagger\phi + 2\rho^2\phi^\dagger\phi A^+A^- - i\rho A^+(\phi^\dagger\partial_+\phi - \phi\,\partial_+\phi^\dagger) + \hat{\delta}\left[\bar{c}(\partial_+A^- + \frac{1}{2}b)\right]\Bigg]. \end{aligned} \tag{28}$$

The last term in the above equation is the extra BRST-invariant gauge-fixing term. After one integration by parts, the above equation could now be written as:

$$\begin{aligned} \mathcal{L}_{BRST} = \Bigg[&\frac{1}{2}(\Pi^-)^2 + \partial_+\phi^\dagger\partial_-\phi + \partial_-\phi^\dagger\partial_+\phi - i\rho A^-(\phi^\dagger\partial_-\phi - \phi\partial_-\phi^\dagger) - \mu^2\phi^\dagger\phi - \frac{\lambda}{6}(\phi^\dagger\phi)^2 \\ &+ 2\rho^2\phi^\dagger\phi A^+A^- - i\rho A^+(\phi^\dagger\partial_+\phi - \phi\,\partial_+\phi^\dagger) + \partial_+A^- + \frac{1}{2}b^2 + (\partial_+\bar{c})(\partial_+c)\Bigg]. \end{aligned} \tag{29}$$

The Euler–Lagrange equation obtained by the variation of $\mathcal{L}_{BRST}$ with respect to $\bar{c}$ implies $\partial_+\partial_+c = 0$. We thus define the bosonic momenta in the usual manner so that $\Pi^+ := b$ but for the fermionic momenta with directional derivatives we set $\Pi_c := \partial_+\bar{c}$ and $\Pi_{\bar{c}} := \partial_+c$, implying that the variable canonically conjugate to $c$ is $\partial_+\bar{c}$ and the variable conjugate to $\bar{c}$ is $\partial_+c$. The quantum BRST Hamiltonian density of the theory is:

$$\begin{aligned} \mathcal{H}_{BRST} = \Bigg[&\frac{1}{2}(\Pi^-)^2 + \Pi^-(\partial_-A^- - 2\rho^2\phi^\dagger\phi A^+A^- + \mu^2\phi^\dagger\phi + \frac{\lambda}{6}(\phi^\dagger\phi)^2 \\ &- i\rho A^-(\phi\,\partial_-\phi^\dagger - \phi^\dagger\partial_-\phi) - \frac{1}{2}(\Pi^+)^2 + \Pi_c\Pi_{\bar{c}}\Bigg]. \end{aligned} \tag{30}$$

The BRST charge operator of the present theory is:

$$Q = \int dx^- \left[ic\partial_-\Pi^- - \rho c(\phi\pi - \phi^\dagger\pi^\dagger) - i\partial_+c\Pi^+\right]. \tag{31}$$

The theory is seen to possess negative norm states in the fermionic sector. The existence of these negative norm states as free states of the fermionic part of $\mathcal{H}_{\mathcal{BRST}}$ is, however, irrelevant to the existence of physical states in the orthogonal subspace of the Hilbert space. The Hamiltonian is also invariant under the anti-BRST transformation given by:

$$\hat{\bar{\delta}}\phi = i\rho\bar{c}\phi, \quad \hat{\bar{\delta}}\phi^\dagger = -i\rho\bar{c}\phi^\dagger, \quad \hat{\bar{\delta}}A^- = -\partial_+\bar{c}, \quad \hat{\bar{\delta}}A^+ = -\partial_-\bar{c}, \tag{32}$$

$$\hat{\bar{\delta}}\pi = \left[-\rho^2\bar{c}\,\phi^\dagger A^+ - i\rho\,\bar{c}\,\partial_-\phi^\dagger\right], \quad \hat{\bar{\delta}}\pi^\dagger = \left[-\rho^2\bar{c}\phi A^+ + i\rho\bar{c}\partial_-\phi\right], \tag{33}$$

$$\hat{\bar{\delta}}u = \hat{\bar{\delta}}v = \hat{\bar{\delta}}w = \hat{\bar{\delta}}\Pi^+ = \hat{\bar{\delta}}\Pi^- = \hat{\bar{\delta}}\Pi_u = \hat{\bar{\delta}}\Pi_v = \hat{\bar{\delta}}\Pi_w = 0, \tag{34}$$

$$\hat{\bar{\delta}}c = -b, \quad \hat{\bar{\delta}}\bar{c} = 0, \quad \hat{\bar{\delta}}b = 0, \tag{35}$$

with generator or anti-BRST charge

$$\bar{Q} = \int dx^- \left[-i\,\bar{c}\,\partial_-\Pi^- - \rho\,\bar{c}\,(\phi\pi - \phi^\dagger\pi^\dagger) + i\,\partial_+\bar{c}\,\Pi^+\right]. \tag{36}$$

We also have

$$\partial_+ Q = [Q, H_{BRST}] = 0, \qquad \partial_+\bar{Q} = [\bar{Q}, H_{BRST}] = 0 \tag{37}$$

with $H_{BRST} = \int dx^- \,\mathcal{H}_{BRST}$, and we further impose the dual condition that both $Q$ and $\bar{Q}$ annihilate physical states, implying that

$$Q|\psi\rangle = 0 \quad \text{and} \quad \bar{Q}|\psi\rangle = 0. \tag{38}$$

The states for which the constraints of the theory hold, satisfy both of these conditions and are in fact, the only states satisfying both of these conditions. Now, because $Q|\psi\rangle = 0$, the set of states annihilated by $Q$ contains not only the set of states for which the constraints of the theory hold but also additional states for which the constraints of the theory do not hold in particular. This situation is, however, easily avoided by additionally imposing on the theory, the dual condition: $Q|\psi\rangle = 0$ and $\bar{Q}|\psi\rangle = 0$. Thus by imposing both of these conditions on the theory simultaneously, one finds that the states for which the constraints of the theory hold satisfy both of these conditions and, in fact, these are the only states satisfying both of these conditions because in view of the conditions on the fermionic variables $c$ and $\bar{c}$ one cannot have simultaneously $c$, $\partial_+ c$ and $\bar{c}$, $\partial_+\bar{c}$ applied to $|\psi\rangle$ to give zero. Thus the only states satisfying $Q|\psi\rangle = 0$ and $\bar{Q}|\psi\rangle = 0$ are those that satisfy the constraints of the theory and they belong to the set of BRST-invariant as well as to the set of anti-BRST-invariant states. Here, the new extended gauge symmetry which replaces the gauge invariance is maintained (even under the BRST gauge-fixing) and hence projecting any state onto the sector of BRST and anti-BRST invariant states yields a theory which is isomorphic to the original GI theory.

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