Relativistic Symmetry in Nuclei

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Abstract

Pseudospin symmetry has been useful in understanding atomic nuclei. We review the arguments that this symmetry is a relativistic symmetry. The condition for this symmetry is that the sum of the vector and scalar potentials in the Dirac Hamiltonian is a constant. We give the generators of pseudospin symmetry. We review predictions that follow from this insight in the origin of pseudospin symmetry. We propose non-relativistic shell model Hamiltonians that have a pseudospin dynamical symmetry. We also derive the exact solutions of the Dirac Hamiltonian for harmonic oscillator potentials in the spin and pseudospin symmetry limits. We also show that there is a higher U(3) or pseudo-U(3) symmetry in each limit, respectively.

Keywords: *Relativistic symmetry; pseudospin symmetry; shell model*

1 Introduction

When I was invited to talk at this meeting celebrating James 70th birthday I was shocked. Shocked because I could not believe that a student of mine was 70 years old and because James looks so young and vital so it was hard to believe he is 70. I arrived at Yale University in 1968 as a young professor. At that time Jim Vary was a graduate student. Jim was ready to do a thesis and he asked me if I could be his advisor. We decided on the topic of two body correlations in lead nuclei. We used a random phase approximation that used two-particle and two hole modes as well as particle-hole modes. We tested the calculations using two nucleon transfer reactions that were being measured at the time at the Yale tandem accelerator and other accelerators around the world [1,2]. Jim received his Ph.D. in 1970. He became a post doc at MIT and his career blossomed ever since and he became a leader in large shell model calculations of nuclear properties.

About the same time a quasi-degeneracy in the one nucleon states of spherical nuclei with quantum numbers $(n\ell_j, n'\ell'_{j'})$ was discovered [3, 4], where $n' = n-1$, $\ell' = \ell+2$, $j' = j+\frac{1}{2}$ and n, ℓ, j are the radial, orbital angular momentum, and total angular momentum quantum numbers, respectively. These quasi-degeneracies persist in recent measurements in nuclei far from stability [5]. The authors realized that, if they define the average of the orbital angular momenta as a pseudo-orbital angular momentum $(\tilde{\ell})$ and then couple a pseudospin $(\tilde{s} = \frac{1}{2})$ to the pseudo-orbital angular momentum, they will get the total angular momenta $(j = \tilde{\ell} \pm \frac{1}{2})$. For example, for the $(1s_{\frac{1}{2}}, 0d_{\frac{3}{2}})$ orbits, $\tilde{\ell} = 1$, which gives the total angular momenta $j = \frac{1}{2}, \frac{3}{2}$. Subsequently pseudospin doublets in deformed nuclei were discovered [6]. Pseudospin symmetry was later revealed to be a symmetry of the Dirac Hamiltonian [7, 8].

Proceedings of International Conference 'Nuclear Theory in the Supercomputing Era — 2013' (NTSE-2013), Ames, IA, USA, May 13–17, 2013. Eds. A. M. Shirokov and A. I. Mazur. Pacific National University, Khabarovsk, Russia, 2014, p. 254.
http://www.ntse-2013.khb.ru/Proc/Ginocchio.pdf.

2 Symmetries of the Dirac Hamiltonian

The Dirac Hamiltonian with a Lorentz scalar potential, $V_S(\vec{r})$, and a potential which is the fourth component of a Lorentz vector potential, $V_V(\vec{r})$, is

$$H = \vec{\alpha}\cdot\vec{p} + \beta\big(V_S(\vec{r}) + M\big) + V_V(\vec{r}), \tag{1}$$

where $\vec{\alpha}$, β are the Dirac matrices, $\vec{p}$ is the momentum, M is the mass, $\vec{r}$ is the radial coordinate, and the velocity of light is set equal to unity, c =1.

2.1 Spin symmetry: A symmetry of the Dirac Hamiltonian

The Dirac Hamiltonian has spin symmetry when the difference of the vector and scalar potentials in the Dirac Hamiltonian is a constant, $V_S(\vec{r}) - V_V(\vec{r}) = C_s$ [9]. Hadrons [10] and anti-nucleons in a nuclear environment have spin symmetry [11]. These are relativistic systems and normally, in such systems, we would expect large spin-orbit splittings, but, in this limit, spin doublets are degenerate. The generators for this SU(2) spin algebra, $\vec{S}$, which commute with the Dirac Hamiltonian with any potential $V(\vec{r})$, spherical or deformed, $[\, H\, ,\, \vec{S}\,] = 0$, are given by [12]

$$\vec{S} = \begin{pmatrix} \vec{s} & 0 \\ 0 & U_p\,\vec{s}\,U_p \end{pmatrix}, \tag{2}$$

where $\vec{s} = \vec{\sigma}/2$ are the usual spin generators, $\vec{\sigma}$ are the Pauli matrices, and $U_p = \frac{\vec{\sigma}\cdot\vec{p}}{p}$ is the helicity unitary operator introduced in [13]. The generators are four by four matrices as appropriate for the Dirac Hamiltonian.

2.2 Pseudospin Symmetry: A Symmetry of the Dirac Hamiltonian

Another SU(2) symmetry of the Dirac Hamiltonian occurs when the sum of the vector and scalar potentials in the Dirac Hamiltonian is a constant, $V_S(\vec{r}) + V_V(\vec{r}) = C_{ps}$ [9]. The generators for this SU(2) algebra $\vec{\tilde{S}}$, which commute with the Dirac Hamiltonian with any potential $V(\vec{r})$, spherical or deformed, $[H, \vec{\tilde{S}}] = 0$, are given by [12]

$$\vec{\tilde{S}} = \begin{pmatrix} U_p\,\vec{s}\,U_p & 0 \\ 0 & \vec{s} \end{pmatrix}. \tag{3}$$

This symmetry was shown to be pseudospin symmetry [7]. The eigenfunctions of the Dirac Hamiltonian in this limit will have degenerate doublets of states, one of which has pseudospin aligned and the other with pseudospin unaligned. The "upper" matrix of the pseudospin generators in Eq. (3), $U_p\,\vec{s}\,U_p$, have the spin intertwined with the momentum which enables the generators to connect the states in the doublet, which differ by two units of angular momentum. The approximate equality in magnitude of the vector and scalar fields in nuclei and their opposite sign have been confirmed in relativistic mean field theories [8] and in QCD sum rules [8, 14].

3 Consequences of relativistic pseudospin symmetry

One immediate consequence of pseudospin symmetry as a relativistic symmetry is that the "lower" matrix of the pseudospin generators in Eq. (3), $\vec{s}$, does not change the radial wavefunction of the "lower" component of the Dirac eigenfunctions. Hence this symmetry predicts that the radial wavefunctions of the "lower" component is the same for the two states in the doublet. Previous to this discovery many relativistic mean field calculations of nuclear properties had been made. Hence this prediction was tested with existing calculations and, indeed, these wavefunctions are very similar for both spherical [15, 16] and deformed nuclei [17, 18]. Because of the momentum dependence of the "upper" matrix of the generators the relationship between the "upper" components involves a differential equation and these have also been tested in spherical [19] and deformed nuclei [18] with success.

Magnetic dipole and Gamow–Teller transitions between the two states in pseudospin doublets are forbidden non-relativistically because the states differ by two units of angular momentum. However, they are not forbidden relativistically which means that they are proportional to the lower component of the Dirac eigenfunction. This leads to a condition between the magnetic moments of the states and the magnetic dipole transition between them because the radial amplitudes of the lower components of the two states in a pseudospin doublet are equal. Therefore the magnetic dipole transition between the two states in the doublet can be predicted if the magnetic moments of the states are known [8, 20]. Likewise pseudospin symmetry also predicts Gamow–Teller transitions between a state in a parent nucleus to the partner state in the daughter nucleus if the Gamow–Teller transition to the same states in the parent and daughter nucleus is known. We do not have space to discuss these relationships in detail but one example occurs in the mirror nuclei ${}^{39}_{19}\mathrm{K}_{20}$ and ${}^{39}_{20}\mathrm{Ca}_{19}$. The ground state and first excited state of ${}^{39}_{19}\mathrm{K}_{20}$ are interpreted as a $0d_{3/2}$ and $1s_{1/2}$ proton hole respectively, while the ground state and first excited state of ${}^{39}_{20}\mathrm{Ca}_{19}$ are interpreted as a $0d_{3/2}$ and $1s_{1/2}$ neutron hole respectively. These states are members of the $\tilde{n}_r = 1$, $\tilde{\ell} = 1$ pseudospin doublet. Using the magnetic moment of ^{39}Ca a transition rate is calculated which is only about 37 % larger than the measured. However, the two states in the doublet are not pure single-particle states. A modification of these relations has been derived which take into account the fact that these states are not pure single particle states [8, 21]. The modified relations give a transition rate that agrees with the measured value to within experimental error. Again using the mass 39 nuclei, the Gamow–Teller transitions from the ground state of ^{39}Ca to the ground and first excited state of ^{39}K are known, which is enough information to predict the transition from the ground state to the excited state. In the non-relativistic shell model an effective tensor term $g_{eff}[Y_2\sigma]^{(1)}$ is added to the magnetic dipole operator and the Gamow–Teller operator to produce a transition, where g_{eff} is a calculated effective coupling constant. However, the magnetic dipole transition calculated between the same states is an order of magnitude lower than the experimental transition [22] although the calculated Gamow–Teller agrees with the experimental value within the limits of experimental and theoretical uncertainty. This inconsistency has been a puzzle for the non-relativistic shell model. On the other hand the relativistic single-nucleon model gives a consistent description of both of these transitions. A global prediction of magnetic dipole transitions throughout the periodic table has had reasonable success as well [8, 21]. However, a global prediction of Gamow–Teller transitions have not been done yet. Pseudospin symmetry can also be used to relate quadrupole transitions between multiplets [8].

4 Anti-nucleon in a nuclear environment

Charge conjugation changes a nucleon into an anti-nucleon. Under charge conjugation the scalar potential of a nucleon remains invariant while the its vector potential changes sign. Hence the pseudospin condition $V_S(\vec{r}) + V_V(\vec{r}) \approx C_{ps}$ becomes $\bar{V}_S(\vec{r}) - \bar{V}_V(\vec{r}) \approx C_{ps} = \bar{C}_s$. Hence approximate pseudospin for nucleons predicts approximate spin symmetry for anti-nucleons in a nuclear environment. Since the potentials are also very deep we would expect approximate $U(3)$ symmetry as well (see Section 6). The sparse data on the scattering of polarized anti-nucleons on nuclei supports this prediction [23]. Perhaps more data on the scattering of polarized antinucleons on nuclei will be forthcoming from GSI.

5 Shell model Hamiltonians with pseudospin as a dynamical symmetry

We would like to go beyond the mean field and use pseudospin symmetry in the non-relativistic shell model. To that goal, we have constructed the most general shell model Hamiltonian with two-nucleon interactions which has pseudospin symmetry and pseudo-orbital angular momentum symmetry as dynamical symmetries. Such Hamiltonians have eigenfunctions with conserved pseudospin and pseudo-orbital angular momentum, but the energy levels are not degenerate. We do not have the space to give the derivation, but we summarize the results.

These shell model Hamiltonians are given in momentum space. The single particle Hamiltonian for A nucleons, $h_{ps} = \Sigma_{k=1}^{A}\, h(\vec{p}_k)$, has the same form as the single particle Hamiltonian that has spin as a dynamical symmetry; that is,

$$h(\vec{p}_k) = h_c(p_k) + h_o(p_k)\, \vec{\ell}_k \cdot \vec{\ell}_k + h_{so}(p_k)\, \vec{s}_k \cdot \vec{\ell}_k. \tag{4}$$

For completely degenerate pseudospin doublets, there is the condition $h_{so}(p_k) = 4h_o(p_k)$ [24].

The two-nucleon interaction, $V_{ps} = \Sigma_{k>t=1}^{A} V(\vec{p}_k, \vec{p}_t)$, is composed of isospin zero and one parts,

$$V(\vec{p}_k, \vec{p}_t) = V^{(0)}(\vec{p}_k, \vec{p}_t)\frac{(1 - \tau_k \cdot \tau_t)}{4} + V^{(1)}(\vec{p}_k, \vec{p}_t)\frac{(3 + \tau_k \cdot \tau_t)}{4}, \tag{5}$$

where

$$\begin{aligned} V^{(T)}&(\vec{p}_k, \vec{p}_t) \\ &= V_c^{(T)} + V_s^{(T)} s_k \cdot s_t + V_o^{(T)} \ell_k \cdot \ell_t + V_{so}^{(T)}(s_k \cdot \ell_t + s_t \cdot \ell_k) \\ &+ V_t^{(T)}\, [s_k s_t]^{(2)} \cdot \Big([\hat{p}_k \hat{p}_k]^{(2)} + [\hat{p}_t \hat{p}_t]^{(2)}\Big) + V_{dt}^{(T)}\Big([s_k s_t]^{(2)} \cdot [\hat{p}_k \hat{p}_t]^{(2)} - [s_k s_t]^{(1)} \cdot [\hat{p}_k \hat{p}_t]^{(1)}\Big) \\ &+ V_{mso}^{(T)}\big([s_k \cdot \hat{p}_k]\, \hat{p}_k \cdot \vec{\ell}_t + [s_t \cdot \hat{p}_t]\, \hat{p}_t \cdot \vec{\ell}_k\big), \end{aligned} \tag{6}$$

and all the coefficients $V_i^{(T)}$ depend on the magnitudes of the momenta, p_k, p_t, and the angle between them, $\theta_{k,t}$. The first line has spin as a dynamical symmetry. However the second line has tensor interactions, and the third line has dipole interactions which break the spin symmetry, but conserve pseudospin symmetry. The tensor interaction has been shown to be important for shell evolution in exotic nuclei [25]. At the same time pseudospin doublets are also seen in these nuclei [5]. Perhaps these shell model Hamiltonians will be able to explain both effects in a unified way.

6 The Dirac Hamiltonian with harmonic oscillator potentials

The Dirac Hamiltonian with harmonic oscillator potentials has been solved analytically in the spin and pseudospin limits [8, 26] but not in the limit of scalar and vector potentials independent of each other. In these two limits there are higher symmetries just as there are for the non-relativistic harmonic oscillators. We shall summarize the results for the spherically symmetric Dirac Hamiltonian.

6.1 The spherically symmetric Dirac Hamiltonian with spin symmetry and harmonic oscillator potentials

The Dirac Hamiltonian for a spherical harmonic oscillator with spin symmetry is

$$H = \vec{\alpha} \cdot \vec{p} + \beta M + (1+\beta)V(r), \tag{7}$$

r is the magnitude of the radial coordinate. The generators for the spin SU(2) algebra are given in Eq. (2). The generators for the orbital angular momentum SU(2) algebra, $\vec{L}$, which commute with the Dirac Hamiltonian with any spherically symmetric potential $V(r)$, $[H\,,\,\vec{L}\,] = 0$, are given by

$$\vec{L} = \begin{pmatrix} \vec{\ell} & 0 \\ 0 & U_p\,\vec{\ell}\,U_p \end{pmatrix}, \tag{8}$$

where $\vec{\ell} = \frac{(\vec{r}\times\vec{p})}{\hbar}$.

6.2 The energy spectrum

With the harmonic oscillator potential $V(r) = \frac{M\omega^2}{2}\,r^2$ the eigenvalue equation is [26]

$$\sqrt{E_N + M}\,(E_N - M) = \sqrt{2\hbar^2\omega^2 M}\left(N + \frac{3}{2}\right), \tag{9}$$

where $N = 2n + \ell$ is the total harmonic oscillator quantum number, n is the radial quantum number and ℓ is the orbital angular momentum. Hence the eigenenergies have the same degeneracies as the non-relativistic harmonic oscillator. This eigenvalue equation is solved with Mathematica,

$$E_N = \frac{M}{3}\left[3B(A_N) + 1 + \frac{4}{3B(A_{n_1,n_2,n_3})}\right], \tag{10}$$

where $B(A_N) = \left(\frac{A_N+\sqrt{A_N^2-\frac{32}{27}}}{2}\right)^{\frac{2}{3}}$, and $A_N = \frac{\sqrt{2}\hbar\omega}{M}\left(N+\frac{3}{2}\right)$. The spectrum is non-linear in contrast to the non-relativistic harmonic oscillator; i. e., the relativistic harmonic oscillator is not harmonic. However for large M, the binding energy goes like

$$E_N - M \approx M\left(\frac{A_N}{\sqrt{2}} + \cdots\right) \approx \hbar\omega\left(N + \frac{3}{2}\right), \tag{11}$$

in agreement with the non-relativistic harmonic oscillator. For small M the spectrum goes as

$$E_N \approx M\Big(A_N^{\frac{2}{3}} + \cdots\Big) \approx M^{\frac{1}{3}}\left[\sqrt{2}\hbar\omega\Big(N + \frac{3}{2}\Big)\right]^{\frac{2}{3}}, \tag{12}$$

which, in lowest order, agrees with the spectrum for $M \to 0$. Hence the harmonic oscillator is not harmonic in the relativistic limit.

6.3 U(3) generators

The relativistic energy spectrum has the same degeneracies as the non-relativistic spectrum [27], even though the dependence on N is different. This suggests that the relativistic harmonic oscillator has a higher U(3) symmetry. The non-relativistic U(3) generators are the orbital angular momentum $\vec{\ell}$, the quadrupole operator $q_m = \frac{1}{\hbar M\omega}\sqrt{\frac{3}{2}}\left(2M^2\omega^2[rr]^{(2)}_m + [pp]^{(2)}_m\right)$, where $[rr]^{(2)}_m$ means coupled to angular momentum rank 2 and projection m, and the total oscillator quantum number operator, $\mathcal{N}_{NR} = \frac{1}{2\sqrt{2}\hbar M\omega}\left(2M^2\omega^2 r^2 + p^2\right) - \frac{3}{2}$. They form the closed U(3) algebra

$$\left[\mathcal{N}_{NR}, \vec{\ell}\,\right] = [\mathcal{N}_{NR}, q_m] = 0, \tag{13}$$

$$\left[\vec{\ell}, \vec{\ell}\,\right]^{(t)} = -\sqrt{2}\,\vec{\ell}\,\delta_{t,1}, \quad \left[\vec{\ell}, q\right]^{(t)} = -\sqrt{6}\,q_m\,\delta_{t,2}, \quad [q,q]^{(t)} = 3\sqrt{10}\,\vec{\ell}\,\delta_{t,1}, \tag{14}$$

with $\mathcal{N}_{NR}$ generating a U(1) algebra whose eigenvalues are the total number of quanta N and $\vec{\ell}$, q_m generating an SU(3) algebra. In the above we use the coupled commutation relation between two tenors, $T_1^{(t_1)}$, $T_2^{(t_2)}$ of rank t_1, t_2, which is $\left[T_1^{(t_1)}, T_2^{(t_2)}\right]^{(t)} = \left[T_1^{(t_1)}, T_2^{(t_2)}\right]^{(t)} - (-1)^{t_1+t_2-t}\left[T_2^{(t_2)}, T_1^{(t_1)}\right]^{(t)}$.

The relativistic orbital angular momentum generators $\vec{L}$ are given in Eq. (8). We shall now determine the the quadrupole operator Q_m and monopole operator $\mathcal{N}$ that commute with the Hamiltonian in Eq. (7). In order for the quadrupole generator

$$Q_m = \begin{pmatrix} (Q_m)_{11} & (Q_m)_{12}\ \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p}\ (Q_m)_{21} & \vec{\sigma}\cdot\vec{p}\ (Q_m)_{22}\ \vec{\sigma}\cdot\vec{p} \end{pmatrix}, \tag{15}$$

to commute with the Hamiltonian, $[Q_m, H] = 0$, the matrix elements must satisfy the conditions,

$$(Q_m)_{12} = (Q_m)_{21}, \tag{16}$$

$$2[(Q_m)_{11}, V] + [(Q_m)_{12}, p^2] = 0, \tag{17}$$

$$2[(Q_m)_{12}, V] + [(Q_m)_{22}, p^2] = 0, \tag{18}$$

$$(Q_m)_{11} = (Q_m)_{12}\,2(V+M) + (Q_m)_{22}\ p^2. \tag{19}$$

One solution is

$$Q_m = \lambda_2 \begin{pmatrix} M\omega^2(M\omega^2\ r^2 + 2M)[rr]^{(2)}_m + [pp]^{(2)}_m & M\omega^2[rr]^{(2)}_m\ \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p}\ M\omega^2[rr]^{(2)}_m & [pp]^{(2)}_m \end{pmatrix}, \tag{20}$$

where λ_2 is an overall constant undetermined by the commutation of Q_m with the Dirac Hamiltonian.

For this quadrupole operator to form a closed algebra, the commutation with itself must be the orbital angular momentum operator as in Eq. (14). This commutation relation gives

$$\begin{aligned}[Q,Q]^{(t)} &= \sqrt{10}\,\lambda_2^2\,M\omega^2\hbar^2 \begin{pmatrix} (M\omega^2 r^2 + 2M)\,\vec{\ell} & \vec{\ell}\ \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p}\ \vec{\ell} & 0 \end{pmatrix} \\ &= \sqrt{10}\,\lambda_2^2\,M\omega^2\hbar^2(H+M)\vec{L}\,\delta_{t,1},\end{aligned} \tag{21}$$

and we get the desired result if $\lambda_2 = \sqrt{\frac{3}{M\omega^2\hbar^2(H+M)}}$. The quadrupole operator then becomes

$$Q_m = \sqrt{\frac{3}{M\omega^2\hbar^2(H+M)}} \begin{pmatrix} M\omega^2(M\omega^2 r^2 + 2M)[rr]^{(2)}_m + [pp]^{(2)}_m & M\omega^2[rr]^{(2)}_m\ \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p}\ M\omega^2[rr]^{(2)}_m & [pp]^{(2)}_m \end{pmatrix}, \tag{22}$$

which reduces to the non-relativistic quadrupole generator for $H \to M$. In the original paper [28] that derives the quadrupole generators there are two typos. In Eq. (6) of that paper, $\frac{M\omega^2}{2}\ r^2$ should be replaced by $M\omega^2 r^2$ and in the non-relativistic quadrupole operator $M^2\omega^2[rr]^{(2)}_m$ should be replaced by $2M^2\omega^2[rr]^{(2)}_m$. Also, the expression for $B(A_N)$ in that paper has a misplaced factor of 2 in the denominator.

For the monopole generator, we can solve the same equations. But there is a simpler way. From Eq. (9) we get,

$$\mathcal{N} = \frac{\sqrt{H+M}(H-M)}{\hbar\sqrt{2M\omega^2}} - \frac{3}{2}. \tag{23}$$

In the non-relativistic limit, $H + M \to 2M$ and the non-relativistic Hamiltonian $(H - M) \to \hbar\omega\big(N + \frac{3}{2}\big)$ which gives the correct result.

The commutation relations are then those of the U(3) algebra,

$$\big[\mathcal{N}, \vec{L}\,\big] = [\mathcal{N}, Q_m] = 0, \tag{24}$$

$$\big[\vec{L}, \vec{L}\,\big]^{(t)} = -\sqrt{2}\vec{L}\,\delta_{t,1}, \quad \big[\vec{L}, Q\big]^{(t)} = -\sqrt{6}Q\,\delta_{t,2}, \quad [Q, Q]^{(t)} = 3\sqrt{10}\vec{L}\,\delta_{t,1}. \tag{25}$$

The spin generators in Eq. (2), $\vec{S}$, commute with the U(3) generators as well as the Dirac Hamiltonian, and so the invariance group is U(3)×SU(2), where the SU(2) is generated by the spin generators, $\big[\vec{S}, \vec{S}\,\big]^{(t)} = -\sqrt{2}\vec{S}\,\delta_{t,1}$.

6.4 The spherically symmetric Dirac Hamiltonian with pseudospin symmetry and harmonic oscillator potential

The Dirac Hamiltonian with pseudospin symmetry is [7]

$$\tilde{H} = \vec{\alpha}\cdot\vec{p} + \beta M + (1-\beta)V(r), \tag{26}$$

which explains the pseudospin doublets observed in nuclei [8]. This pseudospin Hamiltonian can be obtained from the spin Hamiltonian with a transformation

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad M \to -M, \tag{27}$$

which gives the pseudospin and pseudo-orbital angular momentum generators [12]

$$\vec{\tilde{S}} = \begin{pmatrix} U_p\,\vec{s}\,U_p & 0 \\ 0 & \vec{s} \end{pmatrix}, \quad \vec{\tilde{L}} = \begin{pmatrix} U_p\,\vec{\ell}\,U_p & 0 \\ 0 & \vec{\ell} \end{pmatrix}. \tag{28}$$

6.5 Energy spectrum

With the harmonic oscillator potential $V(r) = \frac{M\omega^2}{2}\,r^2$ the eigenvalue equation in the pseudospin limit is [18]

$$\sqrt{E_{\tilde{N}} - M}\,(E_{\tilde{N}} + M) = \sqrt{2\hbar^2\omega^2 M}\left(\tilde{N} + \frac{3}{2}\right), \tag{29}$$

where $\tilde{N} = 2\tilde{n} + \tilde{\ell}$ is the pseudo total harmonic oscillator quantum number, $\tilde{n}$ is the pseudo radial quantum number and $\tilde{\ell}$ is the pseudo-orbital angular momentum. While n is the number of radial nodes and ℓ the rank of the spherical harmonic of the upper Dirac radial amplitude, $\tilde{n}$ is the number of radial nodes and $\tilde{\ell}$ the rank of the spherical harmonic of the lower Dirac radial amplitude. Again the eigenenergies have the same degeneracy pattern as the non-relativistic harmonic oscillator in the spin symmetry limit. This eigenvalue equation is solved on Mathematica,

$$E_{\tilde{N}} = \frac{M}{3}\left[3B(A_{\tilde{N}}) - 1 + \frac{4}{3\,B(A_{\tilde{N}})}\right], \tag{30}$$

where $B(A_N) = \left(\frac{A_{\tilde N}+\sqrt{A_{\tilde N}^2+\frac{32}{27}}}{2}\right)^{\frac{2}{3}}$, and $A_{\tilde N} = \frac{\sqrt{2}\hbar\omega}{M}\left(\tilde N + \frac{3}{2}\right)$. The spectrum is non-linear in contrast to the non-relativistic harmonic oscillator; i. e., the relativistic harmonic oscillator is not harmonic in either limit. Even for small $A_{\tilde N}$ (large M), the binding energy

$$E_{\tilde N} - M \approx M\left(\frac{A_{\tilde N}^2}{4} + \cdots\right), \tag{31}$$

and hence goes quadratically with the the total pseudo-number of quanta and is non-linear even for large M. For large $A_{\tilde N}$ (small M) the spectrum goes as

$$E_{\tilde N} \approx M\left(A_{\tilde N}^{\frac{2}{3}} - \frac{1}{3} + \cdots\right), \tag{32}$$

which, in the lowest order, agrees with the spectrum for spin symmetry.

6.6 Pseudo-U(3) generators

The pseudo-U(3) generators which commute with the Dirac Hamiltonian, $\left[\tilde H, \vec{\tilde S}\right] = \left[\tilde H, \vec{\tilde L}\right] = \left[\tilde H, \tilde Q_m\right] = \left[\tilde H, \tilde{\mathcal N}\right] = 0$, are then obtained by the transformation in Eq. (27) and are given by

$$\tilde Q_m = \sqrt{\frac{3}{M\omega^2\hbar^2(H-M)}}\begin{pmatrix} M\omega^2(M\omega^2 r^2 - 2M)[rr]^{(2)}_m + [pp]^{(2)}_m & M\omega^2[rr]^{(2)}_m\,\vec\sigma\cdot\vec p \\ \vec\sigma\cdot\vec p\, M\omega^2[rr]^{(2)}_m & [pp]^{(2)}_m \end{pmatrix}. \tag{33}$$

The commutation relations are then those of a U(3) algebra,

$$\left[\tilde{\mathcal N}, \vec{\tilde L}\right] = \left[\tilde{\mathcal N}, \tilde Q_m\right] = 0, \tag{34}$$

$$\left[\vec{\tilde L}, \vec{\tilde L}\right]^{(t)} = -\sqrt{2}\vec{\tilde L}\,\delta_{t,1}, \quad \left[\vec{\tilde L}, \tilde Q\right]^{(t)} = -\sqrt{6}\,\tilde Q\,\delta_{t,2}, \quad \left[\tilde Q, \tilde Q\right]^{(t)} = 3\sqrt{10}\vec{\tilde L}\,\delta_{t,1}. \tag{35}$$

However there are no bound Dirac valence states in the pseudospin limit.

Acknowledgments

This research was supported by the US Department of Energy, Contract No. W-7405-ENG-36.

References

[1] J. Vary and J. N. Ginocchio, Nucl. Phys. A **166**, 479 (1971).

[2] J. Vary, R. J. Ascuitto and J. N. Ginocchio, Nucl. Phys. A **185**, 349 (1972).

[3] A. Arima, M. Harvey and K. Shimizu, Phys. Lett. B **30**, 517 (1969).

[4] K. T. Hecht and A. Adler, Nucl. Phys. A **137**, 129 (1969).

[5] O. Sorlin and M. G. Porquet, Progr. Part. Nucl. Phys. **61**, 602 (2008).

[6] A. Bohr, I. Hamamoto and B. R. Mottelson, Phys. Scr. **26**, 267 (1982).

[7] J. N. Ginocchio, Phys. Rev. Lett. **78**, 436 (1997).

[8] J. N. Ginocchio, Phys. Rept. **414**, 165 (2005).

[9] J. S. Bell and H. Ruegg, Nucl. Phys. B **98**, 151 (1997).

[10] P. R. Page, T. Goldman and J. N. Ginocchio, Phys. Rev. Lett. **86**, 204 (2001).

[11] J. N. Ginocchio, Phys. Rept. **315**, 231 (1999).

[12] J. N. Ginocchio and A. Leviatan, Phys. Lett. B **425**, 1 (1998).

[13] A. L. Blokhin, C. Bahri and J. P. Draayer, Phys. Rev. Lett. **74**, 4149 (1995).

[14] T. D. Cohen *et al.*, Progr. Part. Nucl. Phys. **35**, 221 (1995).

[15] J. N. Ginocchio and D. G. Madland, Phys. Rev. C **57**, 1167 (1998).

[16] J. Meng, K. Sugawara-Tanabe, S. Yamaji, P. Ring and A. Arima, Phys. Rev. C **58**, R628 (1998).

[17] K. Sugawara-Tanabe, S. Yamaji and A. Arima, Phys. Rev. C **65**, 054313 (2002).

[18] J. N. Ginocchio, A. Leviatan, J. Meng and S.-G. Zhou, Phys. Rev. C **69**, 03430 (2004).

[19] J. N. Ginocchio, Phys. Rev. C **66**, 064312 (2002).

[20] J. N. Ginocchio, Phys. Rev. C **59**, 2487 (1999).

[21] P. von Neumann-Cosel and J. N. Ginocchio, Phys. Rev. C **62**, 014308 (2000).

[22] E. Hagberg *et al.*, Nucl. Phys. A **571**, 555 (1994).

[23] D. Garetta *et al.*, Phys. Lett. B **151**, 473 (1985).

[24] C. Bahri, J. P. Draayer and S. A. Moszkowski, Phys. Rev. Lett. **68**, 2133 (1992).

[25] T. Otsuka and D. Abe, Progr. Part. Nucl. Phys. **59**, 425 (2007).

[26] J. N. Ginocchio, Phys. Rev. C **69**, 034318 (2004).

[27] J. P. Elliott, Proc. R. Soc. London, Ser. A **245**, 128 (1958).

[28] J. N. Ginocchio, Phys. Rev. Lett. **95**, 252501 (2005).