Light-Front Quantization of Non-Linear Sigma Models

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Abstract

A study of a class of the non-linear sigma models and gauged non-linear sigma models is presented. The canonical structure, constrained dynamics and the instant-form and light-front quantization of these models is reviewed and studied.

Keywords: *Quantum electrodynamics; quantum chromodynamics; non-linear sigma models; instant-form quantization; light-front quantization*

1 Introduction

In this talk, a study of the non-linear sigma models (NLSM) [1–9] and a class of gauged non-linear sigma models (GNLSM) is presented [7, 8]. The canonical structure and constraint quantization [10–18] of these models is studied using the instant-form and light-front dynamics [17,18]. The instant-form (IF) quantization (IFQ) and light-front (LF) quantization (LFQ) of these models is reviewed and studied.

In this talk, we consider a class of non-linear sigma models [1–6] and gauged non-linear sigma models [1–9]. We first review a class of NLSM [1–9] including their canonical structure and constrained dynamics, and then study their IFQ and LFQ [17, 18] using the Hamiltonian [10], path integral [11–13] and Becchi–Rouet–Stora and Tyutin (BRST) [14–16] formulations.

Using the above methods, we study a class of NLSM and GNLSM in one-space one-time dimensions ($2D$). We study their canonical structure and constraint quantization in the IFQ and LFQ, using Dirac's Hamiltonian formulation and the path integral and BRST formulations. Our studies also involve a construction of gauge-invariant (GI) field theories from the gauge-non-invariant (GNI) field theories using the Stueckelberg formalism and other methods. We could recover the physical contents of the original GNI theories from the corresponding newly constructed GI theories under some special non-trivial gauge-fixing conditions (GFC).

A few points about the IF and LF dynamics are in order. In the IF quantization of field theories, one studies the theory on the hyper surfaces defined by the IF time: $t = x^0 = constant$ [17, 18]. On the other hand, in the LFQ [17, 18] of field theories, one studies the theory on the hyper surfaces of the LF defined by the light-cone (LC) time: $\tau = x^+ = (x^0 + x^1)/\sqrt{2} = constant$.

The LFQ [17,18] has several advantages over the IFQ [17,18]. The LF theory, e. g., has more kinematical generators than the corresponding IF theory and the removal of constraints by Dirac's method gives fewer independent dynamical variables in the

[1]Dedicated to Prof. James Vary on his Seventieth Birthday.

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LFQ than in the IFQ. In LFQ there is no conflict with the microcausality principle. In the LFQ of gauge theories, the transverse degrees of freedom of the gauge field can be immediately identified as the dynamical degrees of freedom, as a result, the LFQ remains very economical in displaying the relevant degrees of freedom leading directly to the physical Hilbert space. Also, because the LF coordinates are not related to the conventional instant-form coordinates by a finite Lorentz transformation, the descriptions of the same physical result may be different in IF and LF dynamics. The advantages of the LFQ over the IFQ are reviewed in Ref. [18]. Use of both the IF and LF gives a rather complete dynamics of the system. A study of such theories could be used to test several interesting ideas in field theories.

Further the product of two Fermi fields at the same space-time point is highly singular and leads to regularization ambiguities. In order to take care of these regularization ambiguities one introduces a regularization parameter which appears in the coefficient of the mass term of the U(1) gauge field A^μ. This regularization scheme is often referred to as the standard regularization.

The O(N) nonlinear sigma models in $2D$, where the field sigma is a real N-component field, provide a laboratory for the various nonperturbative techniques, e. g., $1/N$-expansion, operator product expansion, and the low energy theorems. These models are characterized by features like renormalization and asymptotic freedom common with that of quantum chromodynamics and exhibit a nonperturbative particle spectrum, have no intrinsic scale parameter, possess topological charges, and are very crucial in the context of conformal and string field theories where they appear in the classical limit.

The Hamiltonian formulation of the gauge-non-invariant O(N)-NLSM in $2D$ has been studied in Refs. [2, 3, 6] and its two GI versions have been studied in Ref. [6] in the IFQ using the Hamiltonian and BRST formulations. The LFQ of this theory has been studied by us in Ref. [9], using the Hamiltonian, path integral and BRST formulations.

The IFQ of the gauged non-linear sigma model has been studied by us in Ref. [7], and its LFQ has been studied in Ref. [8], using the Hamiltonian and BRST formulations. We now proceed to study these models in some details in the following.

2 The non-linear sigma models

The O(N)-NLSM in one-space one-time dimensions is defined by the action [1–6]:

$$
\begin{aligned}
S &= \int \mathcal{L}(\sigma_k, \lambda)\, d^2x, \\
\mathcal{L} &= \left[\frac{1}{2}\partial_\mu \sigma_k \partial^\mu \sigma_k + \lambda(\sigma_k^2 - 1)\right].
\end{aligned}
\tag{1}
$$

Here $\sigma_k(x,t)$ (with $k = 1,\ 2, \dots,\ N$) is a multiplet of N real scalar fields in two-dimensions, and $\lambda(x,t)$ is another scalar field. The vector field $\sigma(x,t)$ maps the two-dimensional space-time into the N-dimensional internal manifold whose coordinates are $\sigma_k(x,t)$. In the above equation, the first term corresponds to a massless boson (which is equivalent to a massless fermion), and the second term is the usual term involving the nonlinear constraint $(\sigma_k^2 - 1)$ and the auxiliary field. Also $\mu = 0,\ 1$ for the IFQ and $\mu = +,\ -$ for LFQ.

2.1 Instant-form quantization

In the IFQ, the action of the theory reads [1–6]:

$$S = \int \mathcal{L}(\sigma_k, \lambda)\, dx^0 dx^1, \qquad \mathcal{L} = \left[\frac{1}{2}(\partial_0\sigma_k\partial_0\sigma_k - \partial_1\sigma_k\partial_1\sigma_k) + \lambda(\sigma_k^2 - 1)\right]. \tag{2}$$

This model is seen to possess a set of four constraints [1–6]:

$$\begin{aligned} \chi_1 &= p_\lambda \approx 0, \\ \chi_2 &= (\sigma_k^2 - 1) \approx 0, \\ \chi_3 &= 2\sigma_k\pi_k \approx 0, \\ \chi_4 &= \left(2\pi_k^2 + 4\lambda\sigma_k^2 + 2\sigma_k\partial_1\partial_1\sigma_k\right) \approx 0. \end{aligned} \tag{3}$$

Here p_λ and π_k denote the momenta canonically conjugate respectively to λ and σ_k. Also, χ_1 is a Primary constraint and χ_2, χ_3 and χ_4 are the secondary Gauss law constraints. The symbol $\approx$ here denotes a weak equality in the sense of Dirac, and it implies that these constraints hold as strong equalities only on the reduced hypersurface of the constraints and not in the rest of the phase space of the classical theory (and similarly one can consider it as a weak operator equality for the corresponding quantum theory). The canonical Hamiltonian density of the theory is [6]:

$$\mathcal{H}_c = \left[\frac{1}{2}(\pi_k^2 + \partial_1\sigma_k\partial_1\sigma_k) - \lambda(\sigma_k^2 - 1)\right]. \tag{4}$$

After including the primary constraint in the canonical Hamiltonian density with the help of the Lagrange multiplier field u which is treated as a dynamical field, the total Hamiltonian density of the theory is [6]:

$$\mathcal{H}_T = \left[\frac{1}{2}(\pi_k^2 + \partial_1\sigma_k\partial_1\sigma_k) - \lambda(\sigma_k^2 - 1) + p_\lambda u\right]. \tag{5}$$

The Hamilton's equations of motion which preserve the constraints of the theory could now be obtained using this total Hamiltonian density and are omitted here for the sake of brevity [6]. Also the matrix of the Poisson brackets among the constraints χ_i is non-singular. The theory possesses a vector gauge anomaly at the classical level, implying that the theory describes a gauge-non-invariant theory. However, it is possible to construct gauge-invariant models corresponding to this GNI theory using the techniques of constraint quantization. It is also possible to recover the physical contents of the original GNI theory from the newly constructed GI versions. One can also study the instant-form quantization and light-front quantization of these models (cf. Refs. [6] and [9]).

The Dirac quantization procedure in the IF Hamiltonian formulation leads to the non-vanishing equal-time commutation relations for this theory as [2, 6]:

$$\begin{aligned} [\pi_l(x,t), \pi_m(y,t)] &= \frac{-i}{\sigma_k^2}\left[\sigma_l(x)\,\pi_m(y) - \pi_l(x)\,\sigma_m(y)\right]\delta(x-y), \\ [\sigma_l(x,t), \pi_m(y,t)] &= i\left[\delta_{lm} - \frac{\sigma_l(x)\,\sigma_m(y)}{\sigma_k^2}\right]\delta(x-y). \end{aligned} \tag{6}$$

This model is seen to possess a set of (four) second-class constraints implying that it describes a gauge-non-invariant theory. However it is possible to construct gauge-invariant models corresponding to this GNI theory using the techniques of constrained dynamics. One can also recover the physical contents of the original GNI theory from the newly constructed GI versions. Further, it is also possible to study this theory using the LFQ, where the theory becomes GI, as has been done by us in Ref. [9].

In the path integral formulation [11–14], the transition to quantum theory is made by writing the vacuum to vacuum transition amplitude for the theory called the generating functional $Z[J_k]$ for the present theory, in the presence of the external sources J_k it could be written as:

$$Z[J_k] = \int [d\mu]\ \exp\left\{ i \int d^2x\ \left[J_k \Phi^k + p_\lambda \partial_0 \lambda + \pi_k \partial_0 \sigma_k + \Pi_u \partial_0 u - \mathcal{H}_T \right] \right\}. \qquad (7)$$

Here, the phase space variables of the theory are $\Phi^k \equiv (\lambda, \sigma_k, u)$ with the corresponding respective canonical conjugate momenta $\Pi_k \equiv (p_\lambda, \pi_k, \Pi_u)$. The functional measure $[d\mu]$ of the generating functional $Z[J_k]$ is obtained as:

$$\begin{aligned} [d\mu] = {} & \left[16 \sigma_k^2 \sigma_k^2\, \delta(x-y) \right] [d\sigma_k][d\lambda][du][d\pi_k][dp_\lambda] d\Pi_u\ \delta[p_\lambda \approx 0] \delta\left[(\sigma_k^2 - 1) \approx 0 \right] \\ & \times \delta\left[(2\sigma_k \pi_k) \approx 0 \right] \delta\left[(2\pi_k^2 + 4\lambda\sigma_k^2 + 2\sigma_k \partial_1 \partial_1 \sigma_k) \approx 0 \right]. \end{aligned} \qquad (8)$$

2.2 Light-front quantization

In the LFQ the action of this theory reads [9]:

$$\begin{aligned} S &= \int \mathcal{L}(\sigma_k, \lambda)\, dx^+ dx^-, \\ \mathcal{L} &= \left[\partial_+ \sigma_k \partial_- \sigma_k + \lambda(\sigma_k^2 - 1) \right]. \end{aligned} \qquad (9)$$

This model is seen to possess a set of three constraints:

$$\begin{aligned} \psi_1 &= p_\lambda \approx 0, \\ \psi_2 &= (\pi_k - \partial_- \sigma_k) \approx 0, \\ \psi_3 &= (\sigma_k^2 - 1) \approx 0. \end{aligned} \qquad (10)$$

Here p_λ and π_k denote the momenta canonically conjugate respectively to λ and σ_k. Also, ψ_1 and ψ_2 here are primary constraints and ψ_3 is the secondary Gauss law constraint. These constraints form a set of first-class constraints, implying that the theory describes a GI theory. The theory is indeed seen to be invariant under the gauge transformations [9]:

$$\begin{aligned} & \delta\sigma_k = \beta(x^+, x^-), \quad \delta\pi_k = \partial_- \beta(x^+, x^-), \quad \delta v = \partial_+ \beta(x^+, x^-), \\ & \delta\lambda = \delta u = \delta\pi_k = \delta\pi_u = \delta\pi_v = 0, \end{aligned} \qquad (11)$$

where the gauge parameter $\beta \equiv \beta(x^+, x^-)$ is a function of its arguments. The canonical Hamiltonian density of the theory is [9]:

$$\mathcal{H}_c = \left[-\lambda(\sigma_k^2 - 1) \right]. \qquad (12)$$

After including the primary constraints in the canonical Hamiltonian density with the help of the Lagrange multiplier fields u and v which are to be treated as dynamical fields, the total Hamiltonian density of the theory is:

$$\mathcal{H}_T = \left[-\lambda(\sigma_k^2 - 1) + p_\lambda u + (\pi_k - \partial_- \sigma_k) v \right]. \qquad (13)$$

3 Gauge-invariant non-linear sigma models

3.1 Model-A

In Ref. [6], we have constructed and studied a gauge-invariant non-linear sigma model using the Stueckelberg mechanism. In constructing the gauge-invariant model corresponding to the above gauge-non-invariant model, we enlarge the Hilbert space of the theory and introduce a new field θ, called the Stueckelberg field, through the following redefinition of fields [6]:

$$\sigma_k \to \Sigma_k = \sigma_k - \theta, \qquad \lambda \to \Lambda = \lambda + \partial_0\theta. \tag{14}$$

Performing the changes in to the Lagrangian density of the above theory, we obtain the modified Lagrangian density $\mathcal{L}^I$ (ignoring the total space and time derivatives) as [6]

$$\mathcal{L}^I = \mathcal{L} + \mathcal{L}^S \tag{15}$$

with

$$\mathcal{L}^S = \bigg[\frac{1}{2}(\partial_0\theta)^2 - \frac{1}{2}(\partial_1\theta)^2 - \partial_0\sigma_k\,\partial_0\theta + \partial_1\sigma_k\,\partial_1\theta \\ + \partial_0\theta(\sigma_k^2 - 1) - (\lambda + \partial_0\theta)\theta(2\sigma_k - \theta)\bigg], \tag{16}$$

where $\mathcal{L}^S$ is the appropriate Stueckelberg term corresponding to $\mathcal{L}^I$ and in fact, one can easily see that it is possible to recover the physical contents of the original gauge-non-invariant theory under some special gauge choice. This gauge-invariant theory is seen to possess a set of three constraints:

$$\begin{aligned} \eta_1 &= p_\lambda \approx 0, \\ \eta_2 &= \big[\pi_\theta + \pi_k - (\sigma_k^2 - 1) + \theta(2\sigma_k - \theta)\big] \approx 0, \\ \eta_3 &= \big[(\sigma_k^2 - 1) - \theta(2\sigma_k - \theta)\big] \approx 0, \end{aligned} \tag{17}$$

where η_1 and η_2 are primary constraints and η_3 is the secondary Gauss-law constraint of the theory. Here, p_λ, π_θ, π_k are the momenta canonically conjugate respectively to the variables λ, θ and σ_k. The canonical Hamiltonian density of the theory is [6]:

$$\mathcal{H}_c = \bigg[\frac{1}{2}\pi_k^2 + \frac{1}{2}(\partial_1\,\sigma_k)^2 + \frac{1}{2}(\partial_1\theta)^2 - \partial_1\sigma\partial_1\theta - \lambda(\sigma_k^2 - 1) + \lambda\theta(2\sigma_k - \theta)\bigg]. \tag{18}$$

The total Hamiltonian density corresponding to this gauge-invariant theory obtained after including in the canonical Hamiltonian density of the theory the primary constraints of the theory with the help of Lagrange multiplier fields is:

$$\mathcal{H}_T = \bigg[\frac{1}{2}\pi_k^2 + \frac{1}{2}(\partial_1\sigma_k)^2 + \frac{1}{2}(\partial_1\theta)^2 - \partial_1\sigma\,\partial_1\theta - \lambda(\sigma_k^2 - 1) + \lambda\theta(2\sigma_k - \theta) \\ + p_\lambda u + [\pi_\theta + \pi_k - (\sigma_k^2 - 1) + \theta(2\sigma_k - \theta)]v\bigg]. \tag{19}$$

The set of constraints of the theory is first-class, implying that the theory is gauge-invariant. The theory is indeed seen to be invariant under the following gauge-transformations:

$$\begin{aligned} &\delta\sigma_k = \beta(x^0, x^1), \qquad \delta\lambda = -\delta\theta = -\partial_0\beta(x^0, x^1), \\ &\delta p_\lambda = \delta\pi_\theta = \delta\pi_k = 0, \end{aligned} \tag{20}$$

where the gauge parameter $\beta \equiv \beta(x^0, x^1)$ is a function of its arguments. From this gauge-invariant theory, it is however, possible to recover the physical contents of the original gauge-non-invariant theory under some special gauge-fixing conditions. For this we go to a special gauge given by $\theta = 0$, and accordingly choose the gauge-fixing conditions of the theory as [7]:

$$\begin{aligned} \zeta_1 &= (2\sigma_k \pi_k - \pi_\theta - \pi_k) \approx 0, \\ \zeta_2 &= (2\pi_k^2 + 4\lambda\sigma_k^2 + 2\sigma_K + \partial_1\partial_1\sigma_k) \approx 0, \\ \zeta_3 &= \theta \approx 0. \end{aligned} \tag{21}$$

As studied in details in Ref. [6], it is easy to see that the above set of gauge-fixing conditions reproduces precisely the quantum system described by the original gauge-non-invariant theory. The above set of gauge-fixing conditions in fact translates the gauge-invariant version of the theory into the gauge-non-invariant one. The physical Hilbert spaces of the two theories are just the same [6].

3.2 Model-B

In Ref. [6], we have studied another gauge-invariant non-linear sigma model (constructed by Mitra and Rajaraman in Refs. [4, 5], using their procedure of gauge-invariant reformulation). This model is defined by the total Hamiltonian density

$$\mathcal{H}_T = \left[\frac{1}{2}\pi_k^2 + \frac{1}{2}(\partial_1\sigma_k)^2 - \lambda(\sigma_k^2 - 1) + p_\lambda u - \eta(\sigma_k\pi_k)\right], \qquad \eta \equiv \eta(x^\mu) := \left[\frac{\sigma_k\pi_k}{2\sigma_k^2}\right], \tag{22}$$

and its corresponding second-order Lagrangian density [6]

$$\mathcal{L} = \left[\frac{1}{2}\partial_\mu\sigma_k\,\partial^\mu\sigma_k + \lambda(\sigma_k^2 - 1) + \eta(2\sigma_k\,\partial_0\sigma_k\right]. \tag{23}$$

This model possesses a set of two constraints:

$$\begin{aligned} \chi_1 &= p_\lambda \approx 0, \\ \chi_2 &= (\sigma_k^2 - 1) \approx 0. \end{aligned} \tag{24}$$

Here p_λ and π_k denote the momenta canonically conjugate respectively to λ and σ_k. Also, χ_1 is a primary constraint and χ_2 is the secondary Gauss law constraint. Here the remaining two secondary constraints of the original gauge-non-invariant theory have been truncated using the method of Mitra–Rajaraman [7, 8] for the gauge-invariant reformulation of the corresponding original gauge-non-invariant theory. It is important to note here that this method is applicable only to those theories which possess a chain of constraints following from a single constraint. The constraints which have thus been truncated could now be imposed on the original theory as gauge-fixing conditions for the quantization of the gauge-invariant theory under gauge-fixing. The above gauge-invariant theory is indeed seen to be invariant under the gauge-transformations [6]:

$$\delta\eta = \beta(x^0, x^1), \quad \delta\pi_k = 2\sigma_k\beta(x^0, x^1), \quad \delta\lambda = \partial_0\beta(x^0, x^1), \quad \delta\sigma_k = \delta p_\lambda = 0, \tag{25}$$

where the gauge parameter $\beta \equiv \beta(x^0, x^1)$ is a function of its arguments.

4 The gauged non-linear sigma models

In Refs. [7, 8], we have constructed and studied a gauged non-linear sigma model and studied its quantization using the IFQ [7] and LFQ [8]. The GNLSM with the standard regularization in one-space one-time dimensions is defined by the action [7]:

$$S = \int \mathcal{L}(\sigma_k, \lambda, A^\mu)\, d^2x,$$
$$\mathcal{L} = \left[\frac{1}{2}\,\partial_\mu\sigma_k\,\partial^\mu\sigma_k + \lambda(\sigma_k^2 - 1) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA_\mu\,\partial^\mu\sigma_k + \frac{1}{2}ae^2A_\mu A^\mu\right]. \tag{26}$$

In the above equation, the first term corresponds to a massless boson (which is equivalent to a massless fermion), the second term is the usual term involving the nonlinear constraint and the auxiliary field λ, the third term is the kinetic energy term of the electromagnetic vector-gauge field $A^\mu(x,t)$, the fourth term represents the coupling of the sigma field to the electromagnetic field, and the last term is the mass term for the vector gauge field $A^\mu(x,t)$. Here e is the coupling constant that couples the massless fermion(or equivalently the boson) with the U(1) gauge field A^μ. This theory is a well known gauge-invariant theory, possessing a set of first-class constraints. Here we have constructed a gauged version of the usual NLSM by introducing the U(1) gauge field A^μ into the theory. We have also included the mass term for the U(1) gauge-field A^μ into the above Lagrangian, defined by [$\mathcal{L}_m = \frac{1}{2}ae^2A_\mu A^\mu$], where a is the standard regularization parameter. The modified resulting theory then describes the gauged NLSM (GNLSM) with the standard regularization. This theory is seen to be GI and has been studied in details using the IFQ in Ref. [7], and its LFQ has been studied in Ref. [8].

4.1 Instant-form quantization

The GNLSM with the standard regularization is defined by the action (with $\mu, \nu = 0,\ 1$ for IFQ) [7]:

$$S = \int \mathcal{L}(\sigma_k, \lambda, A^\mu)\, d^2x,$$
$$\mathcal{L} = \left[\frac{1}{2}(\partial_0\sigma_k\,\partial_0\sigma_k - \partial_1\sigma_k\,\partial_1\sigma_k) + \lambda(\sigma_k^2 - 1) - e(A_0\partial_0\sigma_k - A_1\partial_1\sigma_k)\right. \tag{27}$$
$$\left. + \frac{1}{2}(\partial_0A_1 - \partial_1A_0)^2 + \frac{1}{2}ae^2(A_0^2 - A_1^2)\right].$$

This model is seen to possess a set of five constraints as follows:

$$\begin{aligned}
\varphi_1 &= \Pi_0 \approx 0,\\
\varphi_2 &= p_\lambda \approx 0,\\
\varphi_3 &= (\partial_1E - e\Pi_k) \approx 0,\\
\varphi_4 &= (\sigma_k^2 - 1) \approx 0,\\
\varphi_5 &= (2\sigma_k\Pi_k + 2eA_0\sigma_k) \approx 0.
\end{aligned} \tag{28}$$

Here p_λ, π_k, Π_0 and $E = \Pi^1$ denote the momenta canonically conjugate respectively to λ, σ_k, A_0 and A_1. Also, φ_1 and φ_2 are primary constraints and φ_3, φ_4 and φ_5 are the secondary Gauss law constraints. Further, these constraints form a set of first-class constraints, implying that the theory possesses the gauge symmetry and is invariant under the following gauge-transformations [7]:

$$\begin{aligned}
&\delta\sigma_k = e\beta(x^0,x^1), \quad \delta\lambda = -\delta A_0 = -\partial_0\beta(x^0,x^1), \quad \delta A_1 = \partial_1\beta(x^0,x^1),\\
&\delta\pi_k = \delta E = \delta\Pi_0 = \delta p_\lambda = \delta\Pi_u = \delta\Pi_v = 0,\\
&\delta u = -\delta v = \partial_0\partial_0\beta(x^0,x^1),
\end{aligned} \tag{29}$$

where the gauge parameter $\beta \equiv \beta(x^0, x^1)$ is a function of its arguments. The canonical Hamiltonian density of the theory is [7]:

$$\mathcal{H}_c = \Big[\frac{1}{2}\big[\pi_k^2 + E^2 + (\partial_1\sigma_k)^2 + e^2 A_0^2\big] - \frac{1}{2}e^2\big(A_0^2 - A_1^2\big) + E\partial_1 A_0 + eA_0\pi_k - eA_1\partial_1\sigma_k - \lambda\big(\sigma_k^2 - 1\big)\Big]. \quad (30)$$

After including the primary constraints in the canonical Hamiltonian density with the help of the Lagrange multiplier fields u and v which are to be treated as dynamical fields, the total Hamiltonian density of the theory is:

$$\mathcal{H}_T = \Big[\frac{1}{2}\big[\pi_k^2 + E^2 + (\partial_1\sigma_k)^2 + e^2 A_0^2\big] + E\partial_1 A_0 + eA_0\pi_k - \frac{1}{2}e^2\big(A_0^2 - A_1^2\big) - eA_1\partial_1\sigma_k - \lambda\big(\sigma_k^2 - 1\big) + \Pi_0 u + p_\lambda v\Big]. \quad (31)$$

4.2 Light-front quantization

The GNLSM with the standard regularization is defined by the action with $\mu, \nu = +, -$ for LFQ) [8]:

$$S = \int \mathcal{L}(\sigma_k, \lambda, A^\mu)\, dx^+ dx^-,$$

$$\mathcal{L} = \Big[(\partial_+\sigma_k)(\partial_-\sigma_k) + \lambda\big(\sigma_k^2 - 1\big) - e\big(A^-\partial_-\sigma_k + A^+\partial_+\sigma_k\big) + \frac{1}{2}\big(\partial_+ A^+ - \partial_- A^-\big)^2 + ae^2\big(A^+ A^-\big)\Big]. \quad (32)$$

This model is seen to possess a set of five constraints:

$$\begin{aligned}
\xi_1 &= \Pi^+ \approx 0,\\
\xi_2 &= p_\lambda \approx 0,\\
\xi_3 &= \big(\pi_k - \partial_-\sigma_k + eA^+\big) \approx 0, \\
\xi_4 &= \big(\partial_-\Pi^- + e\partial_-\sigma_k, + e^2 A^+\big) \approx 0,\\
\xi_5 &= \big(\sigma_k^2 - 1\big) \approx 0.
\end{aligned} \quad (33)$$

Here p_λ, π_k, Π^+ and Π^- denote the momenta canonically conjugate respectively to λ, σ_k, A^- and A^+. Also, ξ_1, ξ_2 and ξ_3 here are primary constraints and ψ_4 and ψ_5 are the secondary Gauss law constraints. These constraints form a set of first-class constraints, implying that the theory describes a GI theory. The theory is indeed seen to be invariant under the gauge transformations [8]:

$$\begin{aligned}
&\delta\sigma_k = e\beta(x^+, x^-), \quad \delta\lambda = -\delta A^- = -\partial_+\beta(x^+, x^-), \quad \delta A^+ = \partial_-\beta(x^+, x^-),\\
&\delta\pi_k = \delta\Pi^+ = \delta\Pi^- = \delta p_\lambda = \delta\pi_k = \delta\Pi_u = \delta\Pi_v = \delta\Pi_w = 0, \\
&\delta u = -\delta v = \partial_+\partial_+\beta(x^+, x^-), \qquad \delta w = e\partial_+\beta(x^+, x^-),
\end{aligned} \quad (34)$$

where the gauge parameter $\beta \equiv \beta(x^0, x^1)$ is a function of its arguments. The canonical Hamiltonian density of the theory is [8]:

$$\mathcal{H}_c = \big[-\lambda(\sigma_k^2 - 1)\big]. \quad (35)$$

After including the primary constraints in the canonical Hamiltonian density with the help of the Lagrange multiplier fields u, v and w which are treated as dynamical fields, the total Hamiltonian density of the theory is:

$$\mathcal{H}_T = \left[-\lambda(\sigma_k^2 - 1) + \Pi^+ u + p_\lambda v + (\pi_k - \partial_- \sigma_k + eA^+)w\right]. \tag{36}$$

Also, in the usual Hamiltonian and path integral formulations of a GI theory under some GFC, one necessarily destroys the gauge invariance of the theory by fixing the gauge (which converts a set of first-class constraints into a set of second-class constraints, implying a breaking of gauge invariance under gauge-fixing). In order to achieve the quantization of a GI theory such that the gauge-invariance of the theory is maintained even under gauge-fixing one goes to a more generalized procedure called BRST formulation.

For the BRST quantization of a GI theory, one enlarges the phase space of the classical theory or the Hilbert space of the corresponding quantum theory of the GI theory by introducing the Faddeev–Popov fermionic ghost and anti-ghost fields and the Nakanishi–Lautrup bosonic ghost field into the first-order Lagrangian density or the action of the theory. One thus rewrites the GI system as a quantum system which possesses a generalized gauge-invariance called the BRST symmetry. In the BRST formulation, one thus embeds a GI theory into a BRST-invariant system, and the quantum Hamiltonian of the system which includes the gauge-fixing contribution commutes with the BRST charge operator Q as well as with the anti-BRST charge operator $\bar{Q}$, and the new symmetry of the quantum system (the BRST symmetry) which replaces the gauge-invariance is maintained even under the gauge-fixing and hence projecting any state onto the sector of BRST and anti-BRST invariant states, yields a theory which is isomorphic to the original gauge-invariant theory.

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