Vertex Functions and Asymptotics of Bound-State Wave Functions

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Contents

The vertex function (VF) W for the virtual n-fragment decay of a bound state a is the matrix element of the process

$$a \rightarrow 1 + 2 + \ldots + n.$$

It is related to the residue in energy of the matrix element of the scattering amplitude of $1 + 2 + \ldots + n \rightarrow 1 + 2 + \cdots + n$.
The matrix element of this process is

$$M = \left\langle \Phi_f | V | P \Psi_i^{(+)} \right\rangle = \left\langle \Phi_f | V | P(1 + GV) \Phi_i \right\rangle, \tag{1}$$

P is the (anti)symmetrization operator. Using the spectral decomposition of the Green function G and the relation $V = H - H_0$ one easily gets

$$W(\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_{n-1}) = -N^{1/2} (T + \epsilon) \langle \phi_1 \phi_2 \ldots \phi_n | \phi_a \rangle. \tag{2}$$

$\vec{q}_i$ - Jacobi momenta, ϕ_i - internal wave functions, T - kinetic energy, ϵ - binding energy in the channel $a \rightarrow 1 + 2 + \ldots + n$.

$N^{1/2}$ arises due to the identity of constituents. If all fragments consist of identical nucleons, then

$$N = \frac{A_a!}{A_1!A_2!\ldots A_n!}. \tag{3}$$

A_i - number of nucleons in i.

If all fragments $1, 2, \ldots n$ are structureless, then $\langle \phi_1 \phi_2 \ldots \phi_n | \phi_a \rangle$ turns into wave function ϕ_a.

VFs W for $a \to 1 + 2 + \ldots + n$ are related to the coordinate asymptotics of ϕ_a in the channel $1 + 2 + \ldots + n$. In what follows we discuss this relation for important cases $n = 2$ and $n = 3$. ($\hbar = c = 1$).

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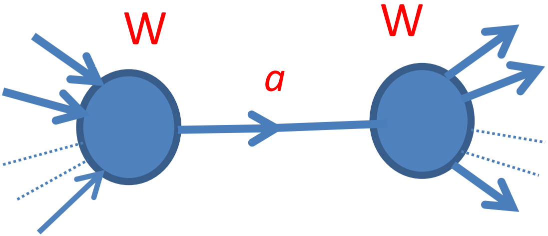

Fig. 1

Two-fragment case ($n = 2$)

From general principles the expression for the vertex function (the matrix element) of two-body decay (virtual or real) $a \to b + c$ can be written as (L.B., I. Borbely, E.I. Dolinskii. Part. Nucl. **8**, 485 (1977)).

$$W_{a\to b+c} = \sqrt{4\pi} \sum_{lsm_l m_s} G_{abc}(ls;\sigma_a,\sigma_b,\sigma_c)$$
$$\times (J_b M_b J_c M_c \mid sm_s)(lm_l sm_s \mid J_a M_a) Y_{lm}(\vec{q}_{bc}/q_{bc}). \quad (4)$$

Here J_i, M_i – spin and its projection of particle i, $(a\alpha b\beta \mid c\gamma)$ – Clebsh-Gordon coefficients, $\vec{q}_{bc}$ – relative momentum of b and c, l and m_l – relative angular momentum of b and c and its projection, s and m_s – channel spin and its projection, Y_{lm} – spherical function, $G_{abc}(ls;\sigma_a,\sigma_b,\sigma_c)$ – invariant vertex form factors (VF). Generally, when all three particles a, b, c are off-shell, G_{abc} may depend on three kinematic invariants and the quantities $\sigma_a, \sigma_b, \sigma_c$ ($\sigma_i = E_i - \vec{p}_i^2/2m_i$) are selected as such invariants in (4). On-shell $\sigma_i = 0$.

However, if one relates a vertex function to the residue of a scattering amplitude and defines it according to (2), then the VF $G_{abc}(ls; \sigma_a, \sigma_b, \sigma_c)$ only depends on relative momenrum q_{bc} which is related to σ_i

$$q_{bc}^2 = -\varkappa^2 - 2\mu_{bc}(\sigma_b + \sigma_c - \sigma_a), \ \varkappa^2 = 2\mu_{bc}\epsilon, \ \epsilon = m_b + m_c - m_a. \quad (5)$$

μ_{ij} is the reduced mass of i and j.

It follows from (5) that if all three particles are on shell ($\sigma_a = \sigma_b = \sigma_c = 0$), then $q_{bc} = i\varkappa$.

On-shell values of VFs are called vertex constants (VC): $G \equiv G(q|_{q=i\varkappa})$. They are analogues of renormalized coupling constants in quantum field theory. VCs thus defined are real.

Using Eq.(4) and the analogous expansion of the overlap function (2) in the coordinate representation one obtains

$$G_{abc}(ls;q) = -(\pi N_{bc})^{1/2}\frac{q^2+\varkappa^2}{\mu_{bc}}\int\limits_0^\infty j_l(qr)I_{abc}(ls;r)r^2dr \qquad (6)$$

where $I_{abc}(ls;r)$ is the radial overlap integral of wave functions of a, b, c, r is the distance between b and c.

VC G is directly related to the asymptotic normalization coefficient (ANC) of $I_{abc}(ls;r)$ at $r \to \infty$. In the case of short-range interaction

$$I_{abc}(ls;r) \approx C_{abc}(ls)\frac{e^{-\varkappa r}}{r}, \quad r \to \infty. \tag{7}$$

Inserting (7) into (6) and putting $q = i\varkappa$, it is easy to obtain the relation between VC $G_{abc}(ls)$ and ANC $C_{abc}(ls)$:

$$G_{abc}(ls) = -\frac{(\pi N_{bc})^{1/2}}{\mu_{bc}} C_{abc}(ls). \tag{8}$$

$N_{bc}^{1/2}$ is often included into the definition of $C_{abc}(ls)$ and $I_{abc}(ls;r)$.

The long-range Coulomb interaction modifies the asymptotic behavior of the overlap integral $I_{abc}(ls;r)$

$$I_{abc}(ls;r) \approx C_{abc}(ls)\frac{W_{-\eta,l+1/2}(\varkappa r)}{r} \approx C_{abc}(ls)\frac{e^{-\varkappa r-\eta\ln(2\varkappa r)}}{r}, \quad r\to\infty. \tag{9}$$

$\eta = Z_b Z_c e^2 \mu_{bc}/\varkappa$ – Coulomb (Sommerfeld) parameter for a bound state a, W – Whittaker function.

In the presence of the Coulomb interaction Eq.(6) could not be used for determining the VC since at $q \to i\varkappa$ the right-hand-side of (6) tends to 0 for repulsive Coulomb potential and to ∞ for attractive potential.

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The total amplitude of elastic bc scattering in the presence of the Coulomb and short-range interactions is written as

$$f(\overrightarrow{k}) = f_C(\overrightarrow{k}) + f_{NC}(\overrightarrow{k}) \quad (10)$$

$$f_C(\overrightarrow{k}) = \sum_{l=0}^{\infty}(2l+1)\frac{\exp(2i\sigma_l)-1}{2ik}P_l(\cos\theta), \quad (11)$$

$$f_{NC}(\overrightarrow{k}) = \sum_{l=0}^{\infty}(2l+1)\exp(2i\sigma_l)\frac{\exp(2i\delta_l^{NC})-1}{2ik}P_l(\cos\theta). \quad (12)$$

Here $\sigma_l = \arg\Gamma(l+1+i\eta_s)$ and δ_l^{NC} – pure Coulomb and Coulomb-nuclear scattering phases, $\Gamma(z)$ – Gamma function, $\eta_s = Z_b Z_c e^2\mu/k$ – Coulomb parameter for a scattering state.

The renormalized Coulomb-nuclear partial-wave amplitude $\tilde{f}_l^N$ is introduced as follows (for repulsive Coulomb potential)

$$\tilde{f}_l^N = \exp(2i\sigma_l)\frac{\exp(2i\delta_l^{NC}) - 1}{2ik}\left(\frac{l!}{\Gamma(l+1+i\eta_s)}\right)^2 e^{\pi\eta_s}. \qquad (13)$$

The analytic properties of $\tilde{f}_l^N$ on the physical sheet are analogous to those for scattering from the short-range potential. In particular, it is regular near zero energy.

If the $b+c$ system possesses the bound state a with the binding energy $\epsilon = \varkappa^2/2\mu$, then the amplitude $\tilde{f}_l^N(k)$ has a pole at $k = i\varkappa$. The residue at that pole is expressed in terms of the Coulomb-renormalized VC $\tilde{G}_l$ and ANC C_l

$$\mathrm{res}\,\tilde{f}_l^N(k) = \lim_{k\to i\varkappa}[(k - i\varkappa)\tilde{f}_l^N(k)] = i\frac{\mu^2}{2\pi\varkappa}\,\tilde{G}_l^2, \qquad (14)$$

$$C_l = -\frac{\mu}{\sqrt{\pi}}\frac{\Gamma(l+1+\eta_0)}{l!}\,\tilde{G}_l, \qquad (15)$$

A knowledge of ANCs is essential for analyzing nuclear reactions between charged particles at low energies. In particular, the value of ANC $C_{abc}(ls)$ determines essentially the cross section of the radiative capture $b(c,\gamma)a$ at astrophysical energies.

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Anomalous asymptotics

In fact, the asymptotic form (7) (or (9)) has been rigorously proved only for the simplest case when the composite system a consists of two elementary constituents. In that case the form (7) follows directly from the Schrödinger equation. It is shown below that the asymptotics of an overlap inegral may differ from eq.(7) if a consists of three or more constituents.

Consider the Fourier transform $J(q^2)$ of $I(r)$

$$I(r) = (2\pi)^{-3} \int e^{i\vec{q}\vec{r}} \, J(q^2) \, d^3q. \tag{16}$$

According to Eqs.(2) and (6) $J(q^2)$ can be written in the form

$$J(q^2) = -N_{bc}^{-1/2} \frac{2\mu_{bc}}{q^2 + \varkappa^2} G(q^2), \tag{17}$$

Inserting eq.(17) into eq.(16) and performing integration over angular variables, one obtains

$$I(r) = \mathrm{const} \cdot \frac{1}{ir} \int_{-\infty}^{\infty} e^{iqr} \frac{G(q^2)}{q^2 + \varkappa^2} q\, dq. \tag{18}$$

In the upper half-plane of the complex variable q the integrand in eq.(18) has a pole at $q = i\varkappa$ and a cut beginning from the nearest singular point $q = i\varkappa_1$ of the form factor $G(q^2)$. Making use of the Cauchy theorem one gets from Eq.(18)

$$I(r) = \mathrm{const} \cdot \{\frac{\pi}{r} e^{-\kappa r} G(-\kappa^2) + \frac{1}{ir} \int\limits_{\kappa_1}^{\infty} \frac{e^{-kr}\,\mathrm{disc}\,G(-k^2)}{k^2 - \kappa^2} kdk\} \tag{19}$$

$$= I_0(r) + I_1(r).$$

The explicit asymptotic form of the second term on the r.h.s. of eq.(19) depends on the behavior of $\mathrm{disc}\, G(q^2)$ at $q^2 \to -\varkappa_1^2$, that is, on the type of the singularity $q = i\varkappa_1$. To investigate the singular behavior of $G(q^2)$, it is convenient to use the formalism of Feynman diagrams. Near the proper singularity $z = z_0$ the singular part of the amplitude of a Feynman diagram having n inner lines and v vertices, behaves as follows

$$\begin{aligned} &M_{nv}\,|_{z\to z_0} \sim (z - z_0)^s \quad \text{if } s \neq 0, 1, 2, ..., \\ &M_{nv}\,|_{z\to z_0} \sim (z - z_0)^s \ln(z - z_0) \quad \text{if } s = 0, 1, 2, ..., \end{aligned} \tag{20}$$

where $s = (3n - 4v + 3)/2$.

The simplest Feynman diagram for an $a \to b + c$ vertex is a triangle diagram of Fig. 2.

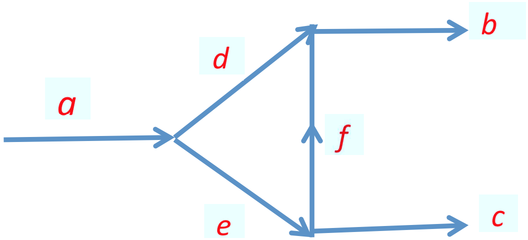

Fig.2

For this diagram $s = 0$ and it is easy to obtain from Eq.(19) that the contribution of that diagram results in

$$I(r)|_{r\to\infty} = c_0 \frac{e^{-\varkappa r}}{r} + c_1 \frac{e^{-\varkappa_1 r}}{r^2}, \qquad (21)$$

$$\varkappa_1 = i\frac{m_b}{m_d}(\varkappa_{ade} + \varkappa_{bdf}),\ \varkappa_{ijk}^2 = 2\mu_{jk}\epsilon_{ijk},\ \epsilon_{ijk} = m_j + m_k - m_i. \qquad (22)$$

The first term on the r.h.s. of Eq.(21) corresponds to the 'normal' asymptotics. If $\varkappa < \varkappa_1$, then this term is a leading one and the overlap integral $I(r)$ possesses the normal asymptotics. However, in the opposite case ($\varkappa > \varkappa_1$) the asymptotics of $I(r)$ is determined by the second term in Eq.(21) (the 'anomalous' case). Though no general rules prevent the 'anomalous' condition $\kappa > \kappa_1$ from being obeyed, it appears that for real nuclear systems this condition is obeyed not very often. The nuclear vertices ${}^{16}\mathrm{O} \to {}^{13}\mathrm{N}({}^{13}\mathrm{C}) + {}^{3}\mathrm{H}({}^{3}\mathrm{He})$ and ${}^{20}\mathrm{Ne} \to {}^{17}\mathrm{F}({}^{17}\mathrm{O}) + {}^{3}\mathrm{H}({}^{3}\mathrm{He})$ can serve as examples of the anomalous asymptotics of the overlap integrals due to the triangle diagram 1.

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Three-fragment case ($n = 3$)

Consider a 3-body bound system $a = \{123\}$ with the wave function

$$\psi_a(\vec{\rho}, \vec{r}), \quad \vec{\rho} = \vec{r}_1 - \vec{r}_2, \quad \vec{r} = \vec{r}_3 - \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}. \tag{23}$$

Constituents 1, 2, and 3 might be composite, then $a = \{123\}$ turns into an overlap integral.

Introduce the Fourier transform $\varphi_a(\vec{k}, \vec{p})$ of $\psi_a(\vec{\rho}, \vec{r})$ and the vertex function (VC) $W(\vec{k}, \vec{p})$

$$\vec{k} = (m_2\vec{k}_1 - m_1\vec{k}_2)/m_{12}, \; \vec{p} = \left[m_{12}\vec{k}_3 - m_3(\vec{k}_1 + \vec{k}_2)\right]/M,$$
$$m_{ij} = m_i + m_j, \; M = m_1 + m_2 + m_3. \tag{24}$$

$$\psi_a(\vec{\rho},\vec{r}) = \int \exp\left[i(\vec{k}\vec{\rho}+\vec{p}\vec{r})\right]\varphi_a(\vec{k},\vec{p})\,\frac{d\vec{k}}{(2\pi)^3}\frac{d\vec{p}}{(2\pi)^3} \tag{25}$$

$$\varphi_a(\vec{k},\vec{p}) = -W(\vec{k},\vec{p})/L(k,p),\; L(k,p) = -(\epsilon + k^2/2\mu_1 + p^2/2\mu_2),$$
$$\epsilon = m_1 + m_2 + m_3 - m_a,\; \mu_1 = m_1 m_2/m_{12},\; \mu_2 = m_1 m_{12}/M. \tag{26}$$

ψ_a and φ_a are normalized

$$\int |\psi_a(\vec{\rho},\vec{r})|^2\, d\vec{\rho}\, d\vec{r} = 1, \qquad \int |\varphi_a(\vec{k},\vec{p})|^2\, d\vec{k}\, d\vec{p}/(2\pi)^6 = 1. \tag{27}$$

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$$\vec{x}_1 = \sqrt{2\mu_1}\vec{\rho},\ \vec{x}_2 = \sqrt{2\mu_2}\vec{r},\ \vec{k}_1 = \vec{k}/\sqrt{2\mu_1},\ \vec{k}_2 = \vec{p}/\sqrt{2\mu_2},$$
$$x_1^2 + x_2^2 = R^2,\ k_1^2 + k_2^2 = P^2. \qquad (28)$$

$\psi_a(\vec{x}_1, \vec{x}_2)$ and $W(\vec{k}_1, \vec{k}_2)$ can be expanded in partial-wave components $\psi_a^{(l,\lambda,L)}(x_1, x_2)$ and $W^{(l,\lambda,L)}(k_1, k_2)$ corresponding to Jacobi angular momenta l and λ ($\vec{l} + \vec{\lambda} = \vec{L}$). Spin variables could be taken into account as well. Strictly speaking, the following text applies to these partial-wave components. However, to simplify the presentation, we suppose that $l = \lambda = 0$ contribute only to ψ_a and W. Then after integrating over angular variables Eq.(25) assumes the form

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$$\psi_a(x_1, x_2) = \frac{(\mu_1\mu_2)^{3/2}}{2\pi^4}\frac{1}{x_1x_2}\int_0^\infty dk_1 \int_0^\infty dk_2 k_1 k_2 \frac{W(k_1, k_2)}{\epsilon + P^2} \times(e^{i\,k_1x_1} - e^{-i\,k_1x_1})(e^{i\,k_2x_2} - e^{-i\,k_2x_2}). \qquad (29)$$

$W(k_1, k_2)$ should depend on k_1^2, k_2^2, that is $W(k_1, k_2)$ is an even function of k_1, k_2. Hence Eq.(29) can be written as

$$\psi_a(x_1, x_2) = \frac{(\mu_1\mu_2)^{3/2}}{2\pi^4}\frac{1}{x_1x_2}\int_{-\infty}^\infty dk_1 \int_{-\infty}^\infty dk_2 k_1 k_2 e^{i(k_1x_1+k_2x_2)} \frac{W(k_1, k_2)}{\epsilon + P^2} \qquad (30)$$

We neglect the Coulomb interaction in what follows though the results obtained could easily be generalized to the case when two of particles 1, 2, and 3 are charged.

If a pair subsystem ij ($ij = 12, 23, 31$) can form a bound state with binding energy ϵ_{ij}, then the VF $W(k_1, k_2)$ has a two-body pole at the relative kinetic energy $E_{ij} = -\epsilon_{ij}$. Such poles lead to the two-body asymptotics analogous to that considered in Section 2. In the present section we will consider the true three-body asymptotics generated by the pole $P^2 = -\epsilon$ in Eq. (30). Denoting that contribution by ψ_3 and integrating over k_2 in the integral (30) by taking the residue at $k_2^2 = \epsilon - k_1^2$, one obtains

$$\psi_3(x_1, x_2) = i\frac{(m_1 m_2 m_3/M)^{3/2}}{2\pi^3}\frac{1}{x_1 x_2}J(x_1 x_2),$$

$$J(x_1 x_2) = \int_{-\infty}^{\infty} dk_1\, k_1 \exp\left(ik_1 x_1 - \sqrt{\epsilon + k_1^2}\, x_2\right) W(k_1, i\sqrt{\epsilon + k_1^2}). \quad (31)$$

Denoting $x_1 = R\cos\alpha,\ x_2 = R\sin\alpha$ one can evaluate $J(x_1 x_2)$ at $R \to \infty$ by the saddle-point method (the saddle-point is $k_1 = i\epsilon^{1/2}\cos\alpha$). As a result, one obtains the following expression for the leading contribution to the asymptotic form of $\psi_3(x_1, x_2)$

$$\psi^{(0)}_{3as}(R,\alpha) = C_3 \, \frac{e^{-\sqrt{\epsilon}R}}{R^{5/2}},$$

$$C_3 = -\frac{(m_1 m_2 m_3/M)^{3/2}}{\sqrt{2}\pi^{5/2}} \, W(i\sqrt{\epsilon}\cos\alpha, i\sqrt{\epsilon}\sin\alpha) \quad (32)$$

The R dependence of the asymptotic form (32) agrees with that presented in S.P.Merkuriev and L.D.Faddeev. Quantum Scattering Theory for Few-Body Systems. Moscow, Nauka, 1985.

C_3 is the 3-body asymptotic normalization factor. It is expressed in terms of the on-shell 3-body vertex function (OSTBVF)

$W(\alpha) \equiv W(i\sqrt{\epsilon}\cos\alpha, i\sqrt{\epsilon}\sin\alpha)$ corresponding to $P^2 = -\epsilon$.

Eq.(32) is the 3-body analogue of the 2-body relation (7).

The saddle-point method allows one to calculate corrections to the leading term (32). In the present work, the expressions for the correction terms of the order $(\sqrt{\epsilon}R)^{-1}$ and $(\sqrt{\epsilon}R)^{-2}$ are obtained. These corrections are expressed in terms of $W(\alpha)$ and its derivatives. The explicit expression for ψ_{3as} including the corrections of the order $(\sqrt{\epsilon}R)^{-1}$ is of the form

$$\psi^{(1)}_{3as} = \psi^{(0)}_{3as}\left[1 + (\sqrt{\epsilon}R)^{-1}\chi_\alpha\right],$$

$$\chi_\alpha = \frac{15}{8} - 2\cot(2\alpha)\,\gamma_1(\alpha) - \frac{1}{2}\gamma_2(\alpha),\ \gamma_n(\alpha) = \frac{d^n W(\alpha)}{d\alpha^n}\,/\,W(\alpha) \quad (33)$$

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The asymptotics of the 3-body wave function was considered in L.B., M.K.Ubaidullaeva, R.Yarmukhamedov. Phys. Atom. Nucl., V.62, P.1289 (1999). The results of that work include the corrections due to non-zero values of the angular momenta l and λ. However, these corrections do not include the terms of the same order due to using the saddle-point method. Making use of the results of the present work, one can calculate the reliable correction terms for $l + \lambda \leq 2$.

Discussion and Conclusion

The on-shell VFs $W(\alpha)$ are important 3-body characteristics determining the asymptotics of 3-body wave functions. Of special interest are quantities $W_0 = W(\alpha = \pi/2)$ corresponding to $k_1 = 0$ what means that particles 1 and 2 move as one body with mass $m_1 + m_2$.
W_0 is a constant which is an analog of the 2-body vertex constant G_{abc}. It could be called the generalized vertex constant (GVC).
It follows from Landau equations that GVCs determine contributions of proper singularities of Feynman diagrams containing loops consisting of two particles (as in Fig.3).

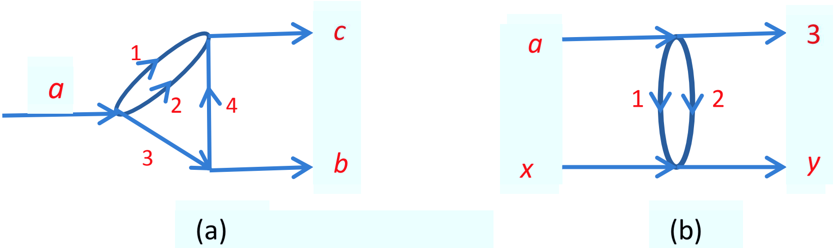

Fig. 3

Thus W_0s $a \to 1+2+3$ and $1+2+4 \to c$ in Fig.3a determine a possible anomalous asymptotics of the overlap integral I^{abc}. W_0s $a \to 1+2+3$ and $x+1+2 \to y$ in Fig.3b determine the contribution of the t-channel normal threshold to the amplitude of the process $a+x \to 3+y$.
The concept of GVC could be directly extended to the loops containing more than 2 particles.

In conclusion it is worthwhile to note that the GVC W_0 for the vertex $a \to 1+2+3$ could be in principle determined by the analytic continuation of the differential cross section of the $a+x \to 1+2+y$ reaction to the pole of the diagram of Fig.4.

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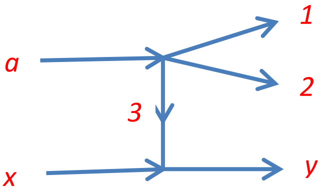

Fig. 4

Thank U 4 attention