BOUNDS ON VARIATION OF THE SPECTRUM AND SPECTRAL SUBSPACES OF A FEW-BODY HAMILTONIAN*

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*Based on joint works with S. Albeverio, V. Kostrykin, K. A. Makarov, and A. V. Selin

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  (in particular to few-body Hamiltonians)

First, we present rather general, abstract results that hold for operators on arbitrary Hilbert spaces. Then we will turn to quantum-mechanical Hamiltonians.

Recalling of the operator norm definition

If V is a bounded linear operator on a Hilbert space $\mathfrak{H}$, its norm $\|V\|$ is given by

$$\|V\| = \sup_{\|f\|=1} \|Vf\| \qquad \text{(N.B.: } \sup = \text{least upper bound).}$$

For any $f \in \mathfrak{H}$ we have $\|Vf\| \leq \|V\|\|f\|$.

If V is a self-adjoint (i.e. Hermitian) operator on $\mathfrak{H}$, and

$$m_V = \min \operatorname{spec}(V) \quad \text{and} \quad M_V = \max \operatorname{spec}(V),$$

then

$$\|V\| = \max\{|m_V|, |M_V|\}.$$

Example 1. $V = |\phi\rangle\kappa\langle\phi|$ with $\|\phi\| = 1$, $\kappa \in \mathbb{R} \Longrightarrow \|V\| = |\kappa|$.

Example 2. $\mathfrak{H} = L_2(\mathbb{R})$, $\quad (Vf)(x) = V(x)f(x)$ with $V(\cdot)$ a bounded function on $\mathbb{R}$. In this case $\|V\| = \sup_{x\in\mathbb{R}} |V(x)|$.

The abstract problem setup

Let A be a self-adjoint operator on a Hilbert space $\mathfrak{H}$ such that

$$\operatorname{spec}(A) = \sigma_0 \cup \sigma_1, \quad \operatorname{dist}(\sigma_0, \sigma_1) = d > 0.$$

The spectral subspaces of A:

$$\mathfrak{H}_0 = \operatorname{Ran} \mathsf{E}_A(\sigma_0), \quad \mathfrak{H}_1 = \operatorname{Ran} \mathsf{E}_A(\sigma_1).$$

A 2×2 operator block matrix representation of A w.r.t. the decomposition $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$:

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}, \quad A_0 = A\big|_{\mathfrak{H}_0}, \quad A_1 = A\big|_{\mathfrak{H}_1}.$$

We focus on the problem of variation of the spectral subspaces under off-diagonal perturbations (i.e. potentials in the case)

$$V = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \qquad (\|V\| = \|B\|).$$

Perturbed operator (total Hamiltonian):

$$H = A + V.$$

Comment: Why off-diagonal perturbations?

One can decompose any bounded V into the sum $V = V_{\rm diag} + V_{\rm off}$ of the diagonal and off-diagonal (w.r.t. $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$) parts

$$V_{\rm diag} = \begin{pmatrix} P_0 V\big|_{\mathfrak{H}_0} & 0 \\ 0 & P_1 V\big|_{\mathfrak{H}_1} \end{pmatrix} \quad \text{and} \quad V_{\rm off} = \begin{pmatrix} 0 & P_0 V\big|_{\mathfrak{H}_1} \\ P_1 V\big|_{\mathfrak{H}_0} & 0 \end{pmatrix},$$

where P_0 and P_1 are the orthogonal projections onto $\mathfrak{H}_0$ and $\mathfrak{H}_1$, respectively, $P_0 = \mathsf{E}_A(\sigma_0)$ and $P_1 = \mathsf{E}_A(\sigma_1)$.

The subspaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$ remain invariant under $V_{\rm diag}$ and, hence, under $A + V_{\rm diag}$. Therefore, for the diagonal perturbations the problem reduces to the perturbation of spectra only.

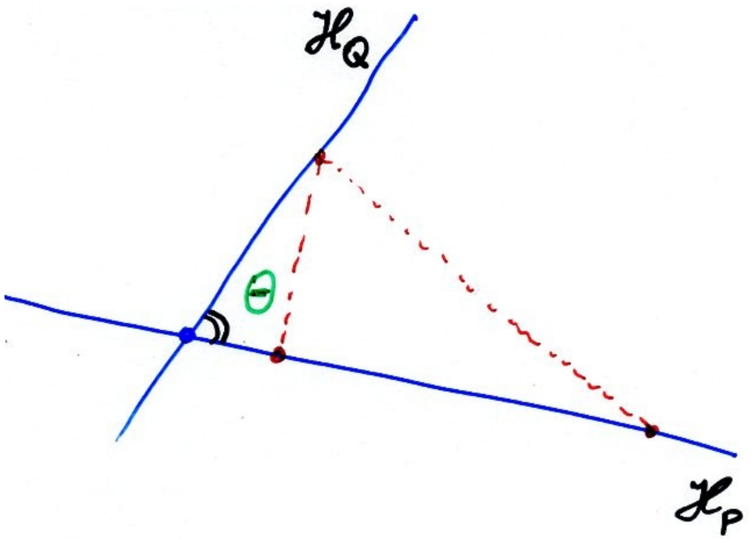
The action of the off-diagonal part $V_{\rm off}$ is completely nontrivial: it may change the spectrum and does change the spectral subspaces. Thus, the *core of the perturbation theory for invariant subspaces is in the study of their variation under off-diagonal perturbations*.

The bounds on variation of the spectral subspaces will be given in terms of the ***maximal angle between two subspaces***.

It is well known that

$$\|P-Q\| \le 1$$

for any two orthogonal projections P and Q in the Hilbert space $\mathfrak{H}$.



Definition. Let $\mathscr{H}_P = \operatorname{Ran} P$ and $\mathscr{H}_Q = \operatorname{Ran} Q$. The quantity

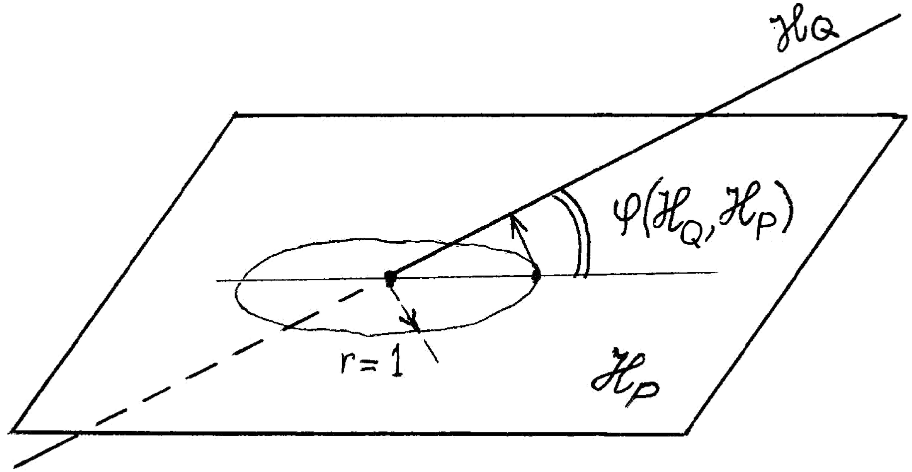
$$\theta(\mathscr{H}_P, \mathscr{H}_Q) := \arcsin(\|P-Q\|)$$

is called the **maximal angle** between the subspaces $\mathscr{H}_P$ and $\mathscr{H}_Q$.

The concept of maximal angle is traced back at least to [Krein, Krasnoselsky, Milman (1948)]; [Dixmier (1949)].

Remark. Assuming that $(\mathscr{H}_P, \mathscr{H}_Q)$ is an **ordered** pair of subspaces in $\mathfrak{H}$ with $\mathscr{H}_P \neq \{0\}$, Krein, Krasnoselsky, and Milman applied the notion of the (relative) maximal angle between $\mathscr{H}_P$ and $\mathscr{H}_Q$ to the number $\varphi(\mathscr{H}_P, \mathscr{H}_Q) \in \left[0, \frac{\pi}{2}\right]$ introduced by

$$\sin\varphi(\mathscr{H}_P, \mathscr{H}_Q) = \sup_{x \in \mathscr{H}_P,\ \|x\|=1} \operatorname{dist}(x, \mathscr{H}_Q).$$



$$\varphi(\mathscr{H}_P, \mathscr{H}_Q) = \frac{\pi}{2}$$

If both $\mathscr{H}_P \neq \{0\}$ and $\mathscr{H}_Q \neq \{0\}$ then

$$\theta(\mathscr{H}_P, \mathscr{H}_Q) = \max\{\varphi(\mathscr{H}_P, \mathscr{H}_Q), \varphi(\mathscr{H}_Q, \mathscr{H}_P)\}.$$

Unlike $\varphi(\mathscr{H}_P, \mathscr{H}_Q)$, the maximal angle $\theta(\mathscr{H}_P, \mathscr{H}_Q)$ is always symmetric w.r.t. the interchange of the arguments $\mathscr{H}_P$ and $\mathscr{H}_Q$.

Furthermore,

$$\varphi(\mathscr{H}_P, \mathscr{H}_Q) = \varphi(\mathscr{H}_Q, \mathscr{H}_P) = \theta(\mathscr{H}_P, \mathscr{H}_Q) \quad \text{whenever} \quad \|P - Q\| < 1.$$

Surely,

$$\|P-Q\| = \sin\big(\theta(\mathscr{H}_P, \mathscr{H}_Q)\big).$$

One is interested in the case where the "rotation angle" from an unperturbed spectral subspace to the perturbed one is acute (i.e. the maximal angle θ between them is smaller than 90°).

Definition. $\mathscr{H}_P$ and $\mathscr{H}_Q$ are in the **acute-angle case** if $\mathscr{H}_P \neq \{0\}$, $\mathscr{H}_Q \neq \{0\}$, and

$$\theta(\mathscr{H}_P, \mathscr{H}_Q) < \frac{\pi}{2},$$

that is, if $\|P-Q\| < 1$.

MAIN QUESTIONS:

(i) What is an optimal requirement on $\|V\|$ ($= \|B\|$) that guarantees that V does not close the gaps between σ_0 and σ_1 (and, thus, $\operatorname{dist}(\sigma_0', \sigma_1') > 0$)?

(ii) What then can be said about variation of the spectral subspace, say, $\mathfrak{H}_0$: Is it then true that the unperturbed and perturbed spectral subspaces $\mathfrak{H}_0$ and $\mathfrak{H}_0'$ are in the acute-angle case, i.e.

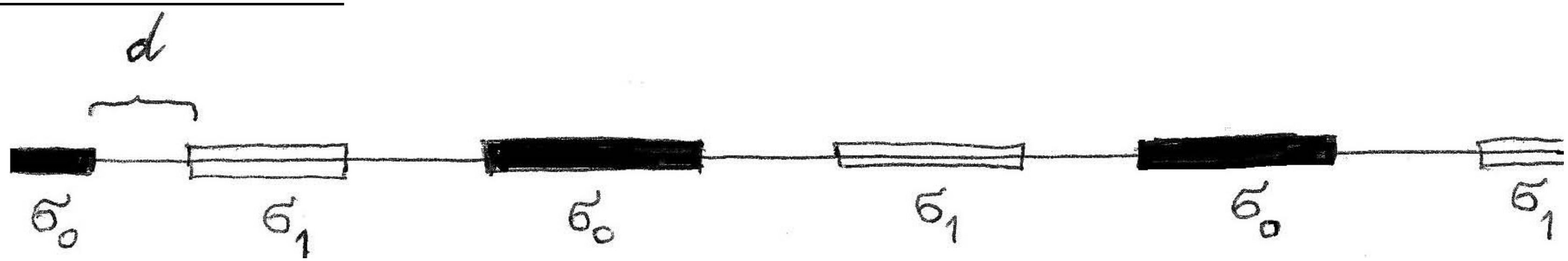
$$\theta(\mathfrak{H}_0, \mathfrak{H}_0') < \frac{\pi}{2}\ ?$$

And what is a (sharp) bound on $\theta := \theta(\mathfrak{H}_0, \mathfrak{H}_0')$ in terms of $\|V\|$ and $d = \operatorname{dist}(\sigma_0, \sigma_1)$?

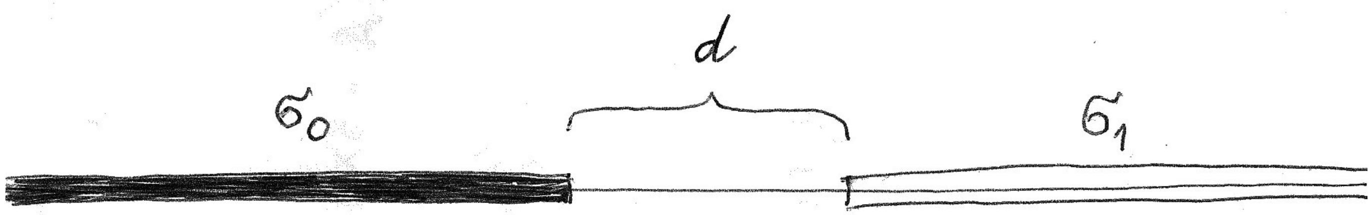
Recall, $\theta(\mathfrak{H}_0, \mathfrak{H}_0')$ stands for the maximal angle between the unperturbed and perturbed spectral subspaces $\mathfrak{H}_0 = \operatorname{Ran} \mathsf{E}_A(\sigma_0)$ and $\mathfrak{H}_0' = \operatorname{Ran} \mathsf{E}_{A+V}(\sigma_0')$.

Under the assumption that $\operatorname{spec}(A) = \sigma_0 \cup \sigma_1$ ***and*** $\sigma_0 \cap \sigma_1 = \varnothing$ ***one distinguishes the following three cases*:**

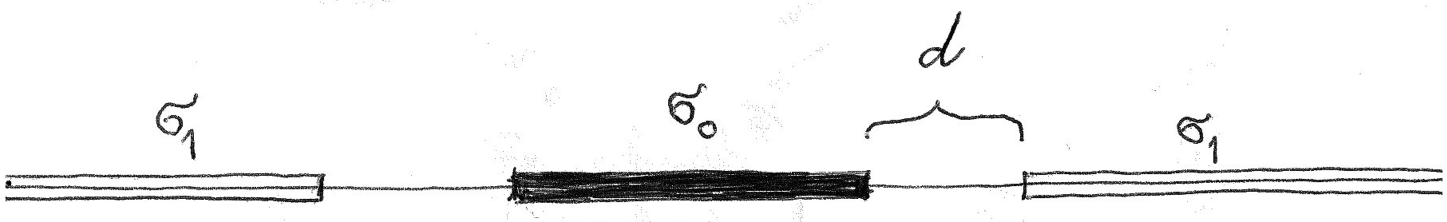
Generic case (**G**): The only condition $\operatorname{dist}(\sigma_0, \sigma_1) = d > 0$.



Special case (**S2**): σ_0 and σ_1 are subordinated, $\operatorname{conv}(\sigma_0) \cap \operatorname{conv}(\sigma_1) = \varnothing$.



Special case (**S3**): One of the sets σ_0 and σ_1 lies in a finite gap of the other one, say $\operatorname{conv}(\sigma_0) \cap \sigma_1 = \varnothing$.



Review of the results for off-diagonal self-adjoint $V=\begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}$

(G) [*V.Kostrykin, K.A.Makarov, A.K.M. (2007)*]: Gaps between σ_0 and σ_1 remain open whenever $\|V\| < \frac{\sqrt{3}}{2} d$ (sharp); $\frac{\sqrt{3}}{2} = 0.866025\ldots$.

$$\theta < \frac{\pi}{2} \quad \text{whenever} \quad \|V\| < c_{\rm MS}\, d, \qquad c_{\rm MS} = 0.675989\ldots$$

[*K.A. Makarov, A. Seelmann (2010, 2013)*]

(S2) For any $\|V\|$ the initial gap between σ_0 and σ_1 remains in $\rho(L)$. The sharp bound for θ:

$$\tan 2\theta \le \frac{2\|V\|}{d} \qquad \Longleftrightarrow \qquad \theta \le \frac{1}{2}\arctan\frac{2\|V\|}{d} \quad \left(< \frac{\pi}{4}\right).$$

(The Davis-Kahan $\tan 2\theta$ Theorem, 1970)

(S3) [*V.Kostrykin, K.A.Makarov, A.K.M. (2005)*]: Gaps between σ_0 and σ_1 remain open and $\theta < \frac{\pi}{2}$ whenever $\|V\| < \sqrt{2}d$ (sharp);

$$\tan\theta \le \frac{\|V\|}{d}$$

[*S. Albeverio, A.V. Selin, A.K.M. (2006, 2012)*]

(see [Integr. Equation Operator Theory **73** (2012), 413]).

Estimates like that in $\tan 2\theta$ Theorem (but in terms of quadratic forms of A and V) have been obtained even for some unbounded V (see [A.K.M., A.V.Selin, Integr. Equations Oper. Theory **56** (2006), 511], [L. Grubišić, V. Kostrykin, K. A. Makarov, K. Veselić, J. Spectr. Theory **3** (2013), 83]).

Bounds on position of the perturbed spectrum

(G) [*V.Kostrykin, K.A.Makarov, A.K.M., 2007*, bounded A], [*C. Tretter, 2009*, unbounded A]:

$$\sigma_i' \subset O_{r_V}(\sigma_i), \quad i = 0, 1,$$

where $O_{r_V}(\sigma_i)$ denotes the closed r_V-neighborhood of σ_i with

$$r_V = \|V\| \tan\left(\frac{1}{2}\arctan\frac{2\|V\|}{d}\right) < \|V\|.$$

(S2) The gap between σ_0 and σ_1 remains in $\rho(A+V)$.

(S3) $\sigma_0' \in O_{r_V}(\sigma_0)$. The gaps between $O_{r_V}(\sigma_0)$ and σ_1 remain in $\rho(A+V)$.

Bounds in the case of non-off-diagonal self-adjoint $V = \begin{pmatrix} V_0 & B \\ B^* & V_1 \end{pmatrix}$

In order to have disjoint perturbed spectral components, one should assume $\|V\| < \frac{d}{2}$. In this case $\sigma_i' \subset O_{\|V\|}(\sigma_i), \quad i = 0, 1$.

(G) The subspaces $\mathfrak{H}_0$ and $\mathfrak{H}_0'$ are in the acute case,

$\theta < \frac{\pi}{2}$, whenever $\|V\| < c_s\, d$, $\quad c_s = 0.454839\ldots$ [*A. Seelmann (2013)*].

In particular,

$$\theta \le \frac{1}{2}\arcsin\frac{\pi\|V\|}{d} < \frac{\pi}{4} \quad \text{if} \quad \|V\| < \frac{1}{\pi} d$$

[*S. Albeverio, A.K.M. (2013)*].

(S2 & S3) The sharp bound for θ (Davis-Kahan $\sin 2\theta$ Theorem, 1970):

$$\theta \le \frac{1}{2}\arcsin\frac{2\|V\|}{d} < \frac{\pi}{4}.$$

Applications to Schrödinger operators (in particular to few-body Hamiltonians)

Let $A = H_0 + V_0$ be the Schrödinger operator with H_0 the kinetic energy and V_0 the "main" potential (combining, say, two-body forces). Let V be an additional interaction (say, three-body forces), and

$$H = A + V.$$

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1. Suppose that E_0 is the g.s. energy (simple eigenvalue) of A, and ψ_0 the g.s. wave function, $A\psi_0 = E_0\psi_0 \quad (\|\psi_0\| = 1)$.

Set $\mathrm{spec}(A) = \sigma_0 \cup \sigma_1$ with $\sigma_0 = \{E_0\}$ and $\sigma_1 = \mathrm{spec}(A) \setminus \{E_0\}$ $(\neq \varnothing)$.

If $2\|V\| < d := \mathrm{dist}(E_0, \sigma_1)$ then there is g.s. (E_0', ψ_0') for H,

$$H\psi_0' = E_0'\psi_0'.$$

We claim that $|\langle \psi_0', \psi_0 \rangle| = \cos\theta$ with $\theta < \pi/4$ such that

$$\sin 2\theta \le \frac{2\|V\|}{d}.$$

This is the consequence of the Davis-Kahan $\sin 2\theta$ Theorem (1970).

If, in addition, V is off-diagonal then, for any arbitrary large $\|V\|$, NO spectrum of H enters the gap between E_0 and σ_1, and

$$|\langle \psi_0', \psi_0 \rangle| = \cos\theta$$

with

$$\tan 2\theta \leq \frac{2\|V\|}{d}.$$

In particular, we have:

$$|\langle \psi_0', \psi_0 \rangle|^2 \geq \frac{1}{2}\left(1 + \frac{d}{\sqrt{d^2 + 4\|V\|^2}}\right) \quad \left(> \frac{1}{2}\right)$$

(Probability of the system to remain in the initial ground state ψ_0.)

Furthermore, necessarily

$$E_0' \leq E_0$$

and

$$E_0 - E_0' \leq \|V\| \tan\left(\frac{1}{2}\arctan\frac{2\|V\|}{d}\right) \quad (< \|V\|).$$

2. Suppose that

$$\sigma_0 = \{E_0, E_1, \ldots, E_n\}$$

consists of the $n+1$ lowest eigenvalues of A and let $\sigma_1 = \mathrm{spec}(A) \setminus \sigma_0$ be the remainder of the spectrum of A.

Denote by $\mathfrak{H}_0$ the spectral subspace of A associated with σ_0, i.e. the linear span of the corresponding eigenvectors.

Assume that $2\|V\| < d := \mathrm{dist}(\sigma_0, \sigma_1)$ and σ_0' combines the eigenvalues of $H = A + V$ that stem from the eigenvalues of A contained in σ_0.

Then

$$\theta \leq \frac{1}{2} \arcsin \frac{2\|V\|}{d} \qquad \left(\Longrightarrow \theta < \frac{\pi}{4} \right),$$

where, recall, $\theta := \theta(\mathfrak{H}_0, \mathfrak{H}_0')$ is the maximal angle between $\mathfrak{H}_0$ and the spectral subspace of $H = A + V$ associated with σ_0'.

If, in addition, V is off-diagonal, then, for any arbitrary large $\|V\|$, NO spectrum of H enters the gap between E_{n+1} $\left(= \max(\sigma_0)\right)$ and σ_1, and

$$\theta \leq \frac{1}{2}\arctan\frac{2\|V\|}{d}.$$

Moreover, necessarily

$$E_0' \leq E_0$$

and

$$E_0 - E_0' \leq \|V\| \tan\left(\frac{1}{2}\arctan\frac{2\|V\|}{d}\right) \quad (< \|V\|).$$

3. Suppose that

$$\sigma_0 = \{E_{n+1}, E_{n+2}, \ldots, E_{n+k}\}, \quad n \geq 0,\ k \geq 1,$$

is a set of consecutive eigenvalues of A and $\sigma_1 = \operatorname{spec}(A) \setminus \sigma_0$.

Notice that there are eigenvalues $E_0, E_1, \ldots, E_n$ of A lying to the left of σ_0. Also there is a part of $\operatorname{spec}(A)$ lying to the right of σ_0.

Assume that

$$\|V\| < \frac{d}{2} \qquad \big(d = \operatorname{dist}(\sigma_0, \sigma_1)\big),$$

and σ_0' consists of the eigenvalues of $H = A + V$ that result from the eigenvalues of A contained in σ_0. Then $\theta = \theta(\mathfrak{H}_0, \mathfrak{H}_0') < \frac{\pi}{4}$ and

$$\sin 2\theta \leq \frac{2\|V\|}{d}.$$

This follows again from the Davis-Kahan $\sin 2\theta$ Theorem.

If V is off-diagonal then the bound may be essentially strengthened:

- The gaps between σ_0 and σ_1 remain open whenever condition

$$\|V\| < \sqrt{2}\, d$$

  is satisfied.

- Moreover, under this condition

$$\tan\theta \leq \frac{\|V\|}{d}.$$

  This is corollary of the a priori $\tan\theta$ Theorem [*S. Albeverio, A.V. Selin, A.K.M. (2006, 2012)*].

Conclusions

- We have found new sharp norm bounds on rotation of spectral subspaces of a self-adjoint operator under off-diagonal perturbations.
- We have also established optimal bounds on the shift of the spectrum under off-diagonal perturbations.
- The maximal angle bounds obtained allow one to derive the corresponding bounds on variation of spectral subspaces under non-off-diagonal (generic) perturbations.
- The general results have been applied to quantum-mechanical (in particular, to few-body) Hamiltonians.
- The spectral shift and subspace variation bounds may be employed to verify the quality of numerical calculations. They may be used to give the corresponding upper estimates prior the actual calculations.

Ideas of the proof, e.g., of the a priori $\tan\theta$ *theorem: Relation to the operator Riccati equation*

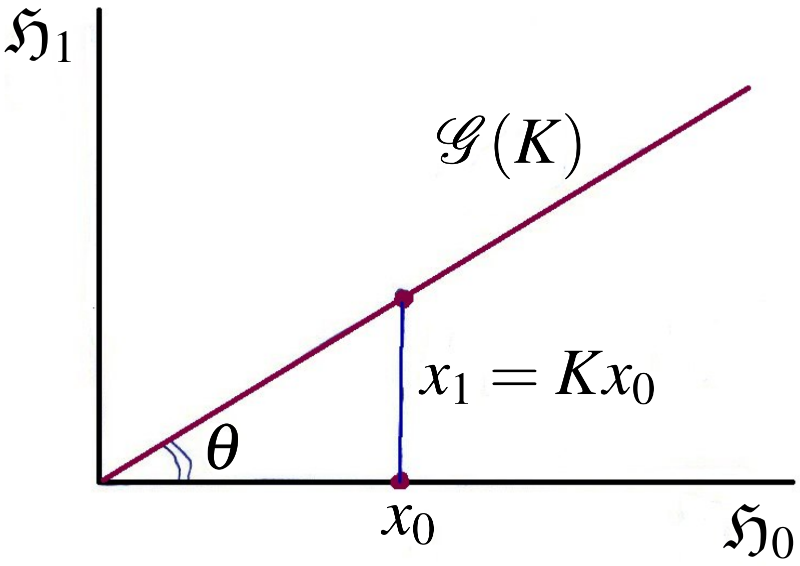
Let $K:\mathfrak{H}_0\to\mathfrak{H}_1$ be a bounded operator.

The graph of K (the graph subspace associated with K)

$$\mathscr{G}(K)=\left\{\begin{pmatrix} x\\ Kx\end{pmatrix}\,\middle|\; x\in\mathfrak{H}_0\right\}$$

is an invariant subspace for $H=\begin{pmatrix} A_0 & B\\ B^* & A_1\end{pmatrix}$ *if and only if* K *is a solution to the operator Riccati equation*

$$KA_0-A_1K+KBK=B^*. \qquad (R)$$



The maximal angle θ between $\mathfrak{H}_0$ and $\mathscr{G}(K)$ is given by

$$\tan\theta=\|K\|.$$

Proposition. Let P and Q are orthogonal projections in $\mathfrak{H}$ with $\mathscr{H}_P = \operatorname{Ran} P$ and $\mathscr{H}_Q = \operatorname{Ran} Q$. Then

$$\|P-Q\| < 1 \Longleftrightarrow \mathscr{H}_Q = \mathscr{G}(K) \qquad \left(\|K\| = \frac{\|P-Q\|}{\sqrt{1-\|P-Q\|^2}} \right).$$

for some bounded operator K from $\mathscr{H}_P$ to $\mathscr{H}_P^{\perp} = \mathfrak{H} \ominus \mathfrak{H}_P$.

Remark. $\mathscr{G}(K)^{\perp} = \mathscr{G}(-K^*)$.

Theorem. *If the graph subspace $\mathscr{G}(K)$, $K \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$, is an invariant subspace for $H = \begin{pmatrix} A_0 & B \\ B^* & A_1 \end{pmatrix}$ then*

$$H = U \Lambda U^*,$$

where U is a unitary operator on $\mathfrak{H}$ given by

$$U = \begin{pmatrix} I & -K^* \\ K & I \end{pmatrix} \begin{pmatrix} I + K^*K & 0 \\ 0 & I + KK^* \end{pmatrix}^{-1/2}$$

and Λ is a block diagonal self-adjoint operator on $\mathfrak{H}$,

$$\Lambda = \operatorname{diag}(\Lambda_0, \Lambda_1),$$

whose entries

$$\Lambda_0 = (I + K^*K)^{1/2}(A_0 + BK)(I + K^*K)^{-1/2}$$

and

$$\Lambda_1 = (I + KK^*)^{1/2}(A_1 - B^*K^*)(I + KK^*)^{-1/2}$$

are self-adjoint operators on the Hilbert spaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$, resp.

In case (**S3**) the existence of the corresponding (in certain sense unique) bounded solution $K:\mathfrak{H}_0\to\mathfrak{H}_1$ to the operator Riccati equation under condition $\|V\|<\sqrt{2}d$ has been proven by Kostrykin, Makarov, A.K.M. (2005) (based on the Virozub-Matsaev factorization theorem).

Polar decomposition of K:

$$K=U|K|,$$

with U the isometry on $\operatorname{Ran}(|K|)=\operatorname{Ran}(K^*)$; $U:\operatorname{Ran}(K^*)\to\operatorname{Ran}(K)$.

Our first idea is to obtain an estimate for eigenvalues of $|K|$ (if they exist).

Lemma. *Let K be a bounded solution to the operator Riccati equation*

$$KA_0-A_1K+KBK=B^*$$

(with $B\neq 0$). Suppose that $|K|$ has an eigenvalue $\lambda>0$, $|K|u=\lambda u$ for some $u\in\mathfrak{H}_0$, $\|u\|=1$. Then the following identity holds:

$$\lambda^2\big(\|A_1Uu\|^2+\|BUu\|^2-\|\Lambda_0u\|^2\big)=\|A_0u\|^2+\|B^*u\|^2-\|\Lambda_0u\|^2,$$

where $\Lambda_0=(I+K^*K)^{1/2}(A_0+BK)(I+K^*K)^{-1/2}$.

In case (S3) we appropriately choose the origin of the spectral parameter plane and, under condition $\|V\| < \sqrt{2}d$, notice that

$$\|A_1 U u\|^2 + \|BU u\|^2 - \|\Lambda_0 u\|^2 > 0.$$

Then the above identity transforms into

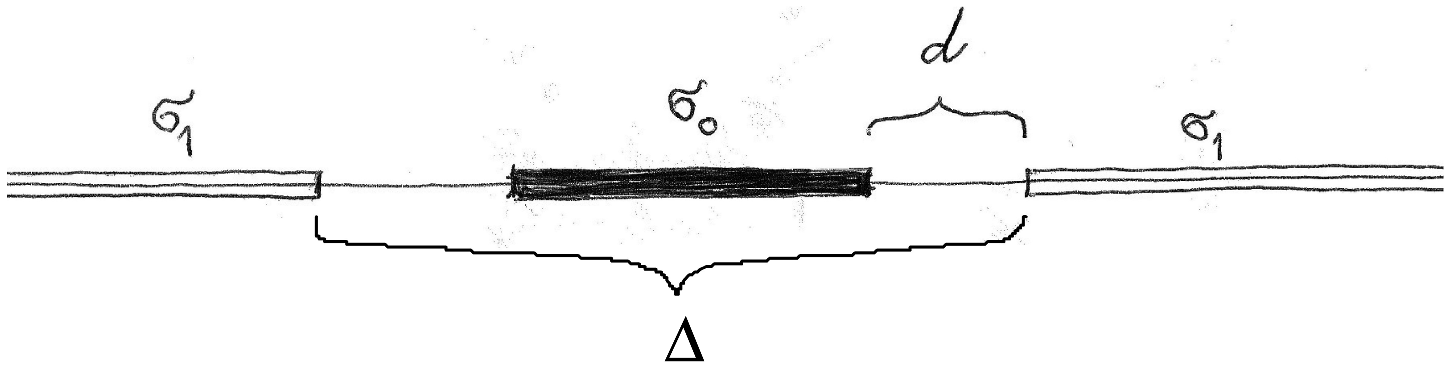
$$\lambda^2 = \frac{\|A_0 u\|^2 + \|B^* u\|^2 - \|\Lambda_0 u\|^2}{\|A_1 U u\|^2 + \|BU u\|^2 - \|\Lambda_0 u\|^2}. \qquad (\star)$$

If $\mathfrak{H}_0$ is finite dimensional then K is finite rank and the equality $(\star)$ is used to find a bound for the maximal eigenvalue of $|K|$, that is, a bound for the norm of K,

$$\|K\| \leq \frac{\|V\|}{d} \quad \Longleftrightarrow \quad \tan\theta \leq \frac{\|V\|}{d}.$$

Further on, by using the result for the finite-rank case, we prove this bound for the infinite-dimensional case.

To be more precise, our complete consideration involves additional parameter, the length $|\Delta|$ of the open gap Δ of σ_1 that contains the whole set σ_0. Our detail estimates for θ, thus, involve three parameters: $\|V\|$, d, and $|\Delta|$.



(Δ is the gap of σ_1 that contains the whole set σ_0)

Under the (**sharp**) gap-nonclosing condition

$$\|V\| < \sqrt{d|\Delta|}$$

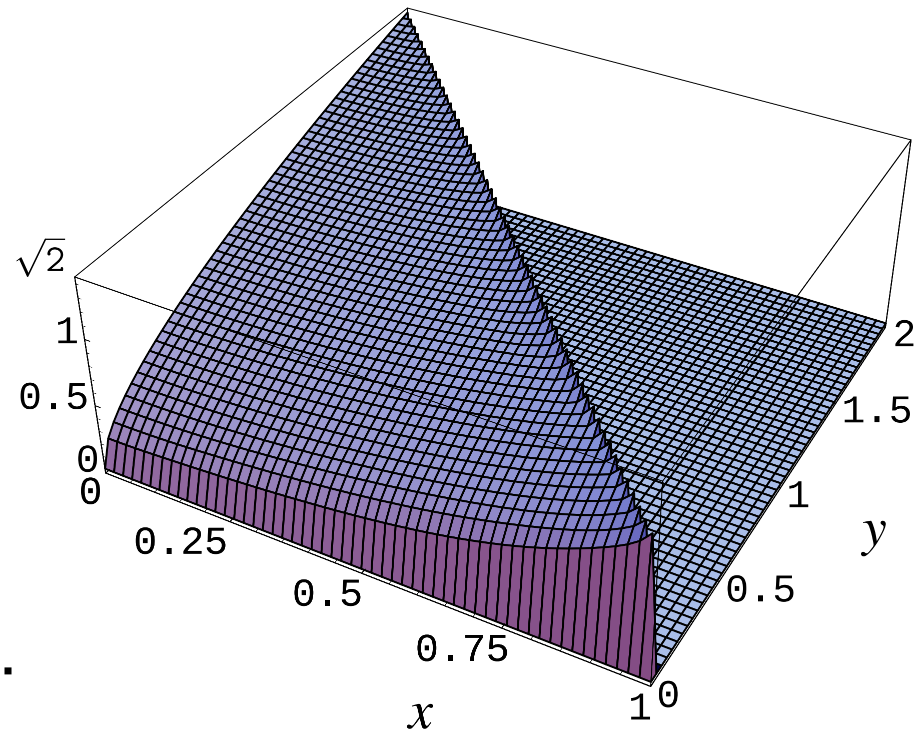
there is an optimal estimating function $M(D,d,v)$, defined for

$$d > 0,\ D \geq 2d,\ 0 \leq v < \sqrt{dD},$$

such that

$$\tan\theta \leq M(|\Delta|,d,\|V\|) \quad (< \sqrt{2}).$$

Explicit expression for M has been found.



For $\|V\| < \sqrt{2}d$,

$$\sup_{D \geq 2d} M(D,d,\|V\|) = \frac{\|V\|}{d}.$$

The **sharp** estimating function $M(D,d,v)$ is plotted in the figure above right, in terms of the “dimensionless” variables

$$x := \frac{D-2d}{D} \quad \text{and} \quad y := \frac{4v^2}{D^2} \qquad \left[\, 0 \leq x < 1, \quad 0 \leq y < 2(1-x) \,\right].$$

Reference: S. Albeverio and A. K. Motovilov, *The a priori* Tan Θ *Theorem for spectral subspaces*, Int. Eq. Oper. Theory, **73** (2012), 413–430; arXiv:1012.1569.