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Bound states of relativistic nature

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• Schrödinger equation with Coulomb potential

$$\Delta\psi + \frac{2m}{\hbar^2}[E - V(r)] = 0,$$

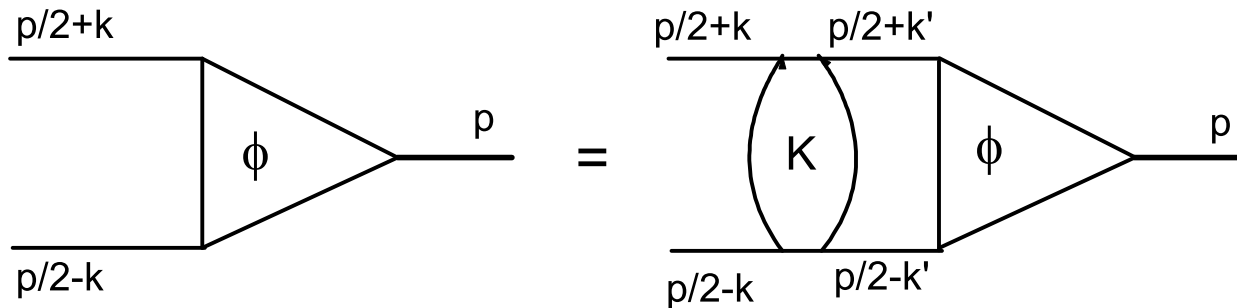
$$V(r) = -\frac{\alpha}{r}$$

System of two particles, each has mass m .
Binding energies (well-known solution):

$$E_n = -\frac{\alpha^2 m}{4n^2} \quad \text{– Balmer series.}$$

● Bethe-Salpeter bound state equation

E.E. Salpeter, H. Bethe, 1951



$$\Phi(k, p) = \frac{ig^2}{\left(\left(\frac{p}{2} + k\right)^2 - m^2 + i\epsilon\right) \left(\left(\frac{p}{2} - k\right)^2 - m^2 + i\epsilon\right)} \times \int \frac{d^4 k'}{(2\pi)^4} \frac{\Phi(k', p)}{[(k - k')^2 - \mu^2 + i\epsilon]}, \quad \mu = 0$$

• Solving BS equation

Nakanishi, 1963

Integral representation for the BS amplitude:

$$\Phi(k, p) = \int_{-1}^1 dz' \int_0^\infty d\gamma' \frac{g(\gamma', z')}{[\gamma' + m^2 - \frac{1}{4}M^2 - k^2 - p \cdot k z' - i\epsilon]^3}.$$

For massless exchange:

$$g(\gamma, z) \rightarrow g(z)\delta(\gamma) \rightarrow g(z)$$

– does not depend on γ .

● "Abnormal" solutions

In 1954, G.C. Wick and R.E. Cutkosky,
still for massless exchange $\mu = 0$,

solved BS equation and reproduced Balmer series.

In addition, they found another series,
which is absent in the Schrödinger equation.

These new solutions, which do not exist in the
non-relativistic limit, were called
"abnormal" solutions.

• Equation for $g(z)$

$\mu = 0 \rightarrow$ G.C. Wick, R.E. Cutkosky, 1954

$$g_n''(z) + \frac{2(n-1)z}{(1-z^2)} g_n'(z) - \frac{n(n-1)}{(1-z^2)} g_n(z) + \frac{\alpha}{\pi} \frac{1}{(1-z^2)(1-\eta^2 + \eta^2 z^2)} g_n(z) = 0$$
$$\eta = \frac{M}{2m} = 1 - \frac{B}{2m}, \quad g_n(\pm 1) = 0$$

For each n – there is infinite set of solutions which can be enumerated by another quantum number: $k = 0, 1, 2, 3, \dots$

● Spectrum

In general: $E = E_{nk}$, $n = 1, 2, 3, \dots$, $k = 0, 1, 2, 3, \dots$

If $k = 0$, the **normal** Balmer series is reproduced
(with a relativistic correction):

$$E_n = -\frac{\tilde{\alpha}^2 m}{4n^2}, \quad \alpha^2 \rightarrow \tilde{\alpha}^2 = \alpha^2 \left(1 + \frac{4}{\pi} \alpha \log \alpha \right)$$

However, for each given n there exists another (**abnormal**)
series with $k = 1, 2, 3, \dots$:

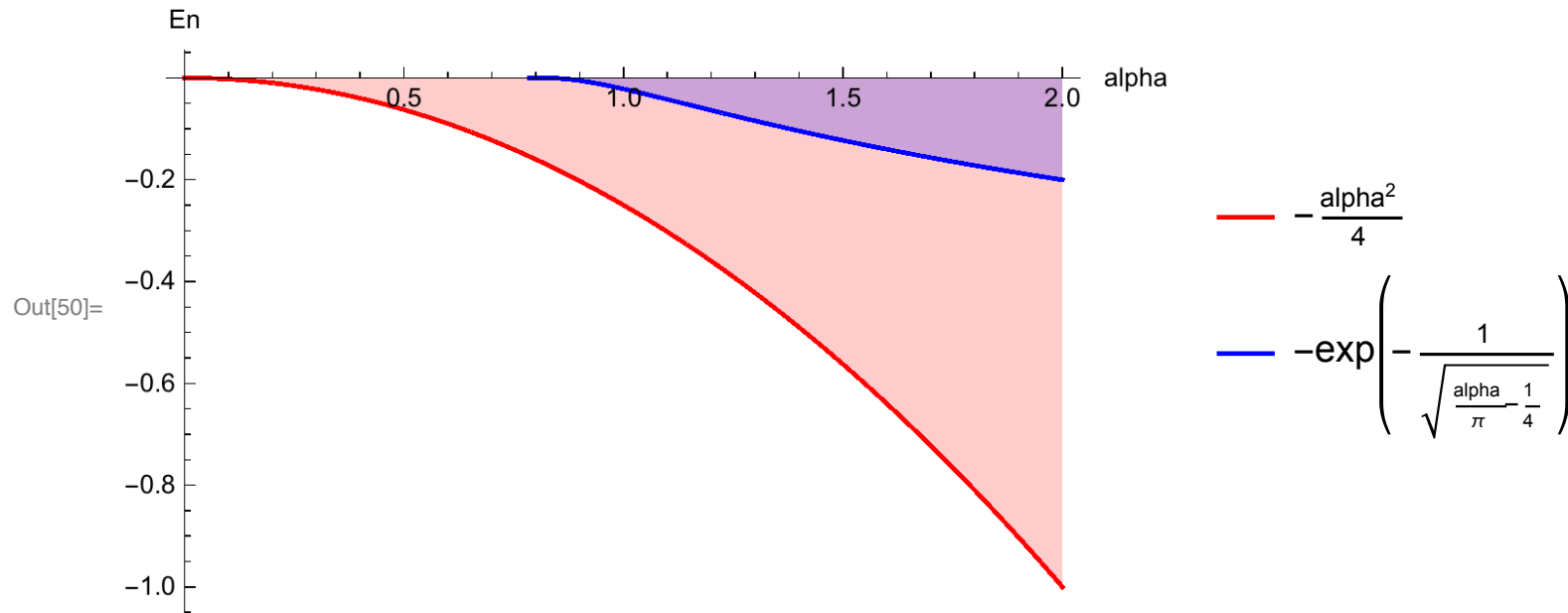
$$E_k = -m \exp \left(-\frac{2\pi^{3/2} k}{\sqrt{\alpha - \frac{\pi}{4}}} \right), \quad k = 1, 2, 3, \dots, \quad \alpha > \frac{\pi}{4}.$$

k is number of nodes in the (abnormal) solution $g_k(z)$.

This formula is valid when $\alpha \rightarrow \frac{\pi}{4}$, $E_k \rightarrow 0$.

$\alpha = e^2/(\hbar c) \rightarrow \alpha/c \rightarrow 0$ when $c \rightarrow \infty$

● Energy spectrum (still for $\mu = 0$)



The binding energies for **normal** and **abnormal** states.

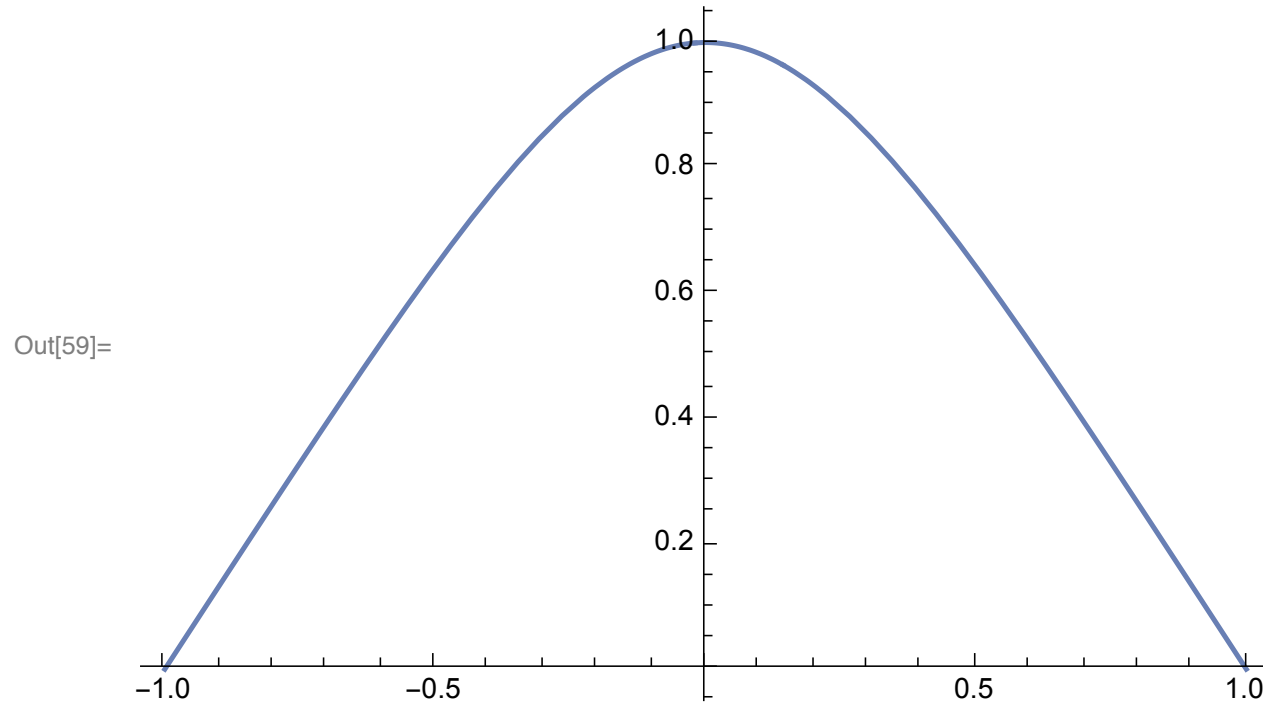
Abnormal states are not predicted by the Schrödinger equation, but they are predicted by the BS one! Therefore they have purely relativistic origin.

Normal solutions $g(z)$ vs. z have no nodes,
abnormal solutions have k nodes.

For even k they are symmetric,
 for odd k they are antisymmetric.

• Spectral function $g(z)$

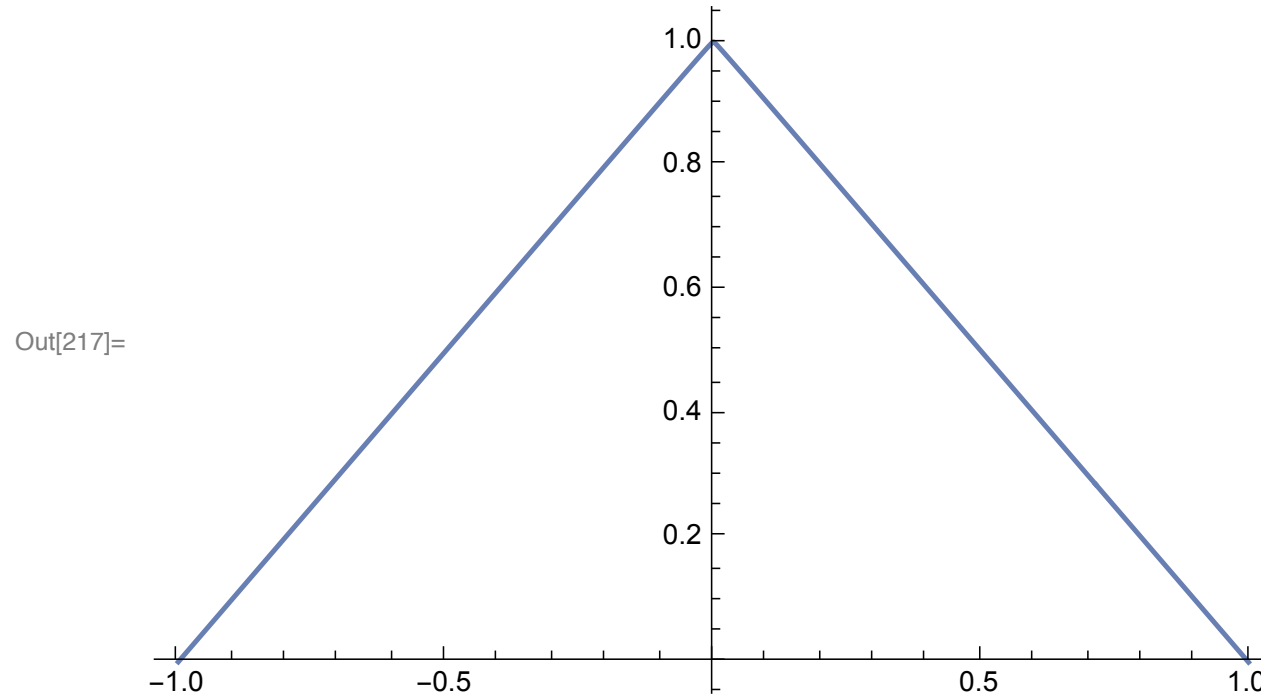
Normal (ground) state, $n = 1, k = 0$



$$B = 0.2, \alpha = 1.786$$

• Spectral function $g(z)$

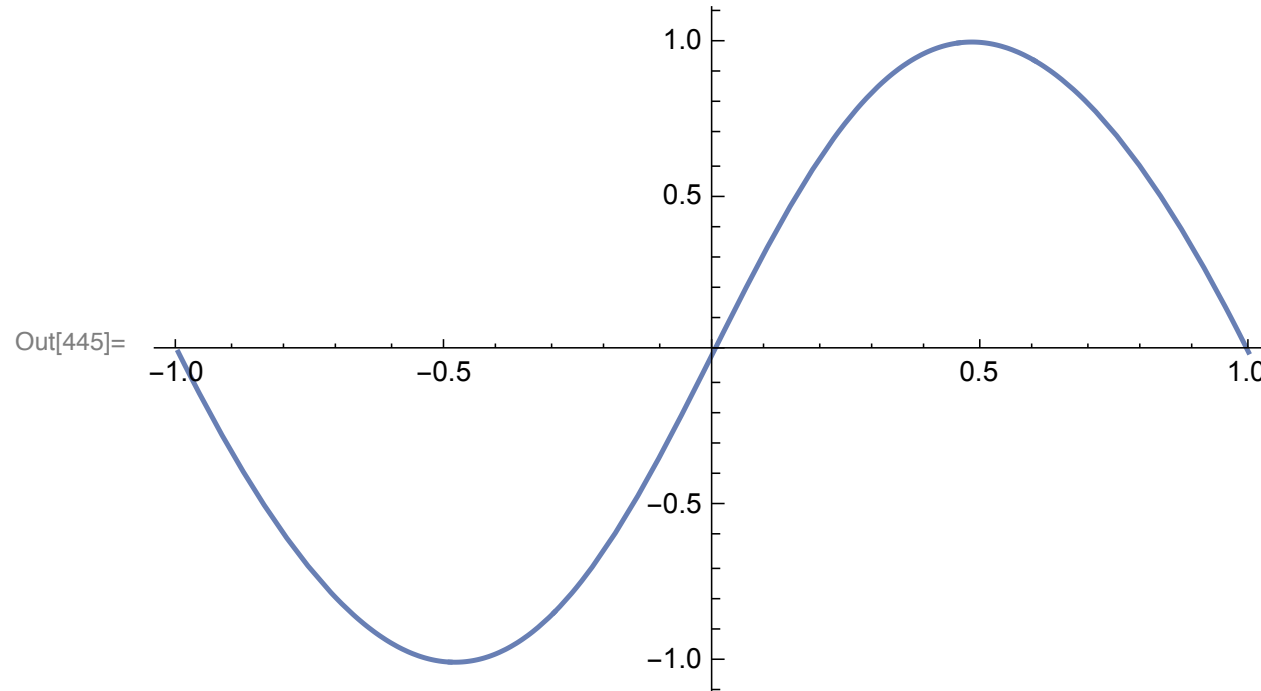
Normal (ground) state, small binding energy, $n = 1$, $k = 0$



$$B = 10^{-6}, \alpha = 0.0020169, g(z) = 1 - |z|$$

• Spectral function $g(z)$

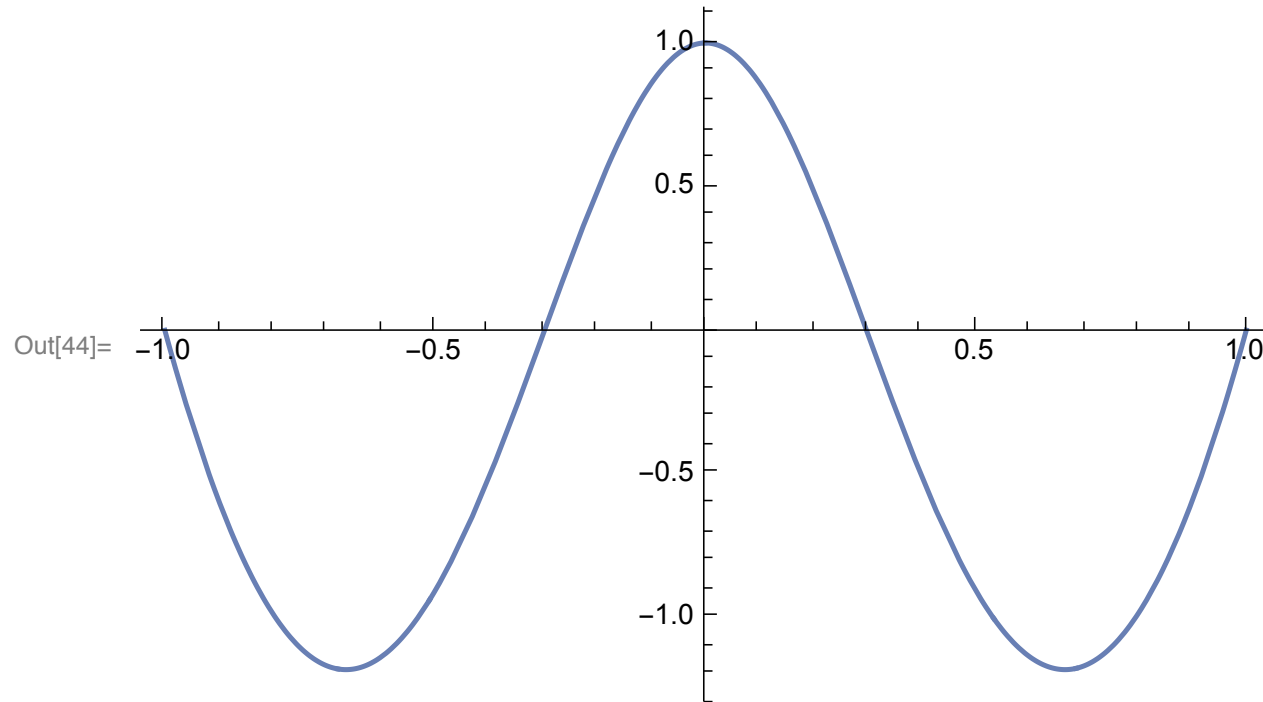
One node, abnormal state, antisymmetric, $n = 1$, $k = 1$



$$B = 0.2, \alpha = 8.25$$

• Spectral function $g(z)$

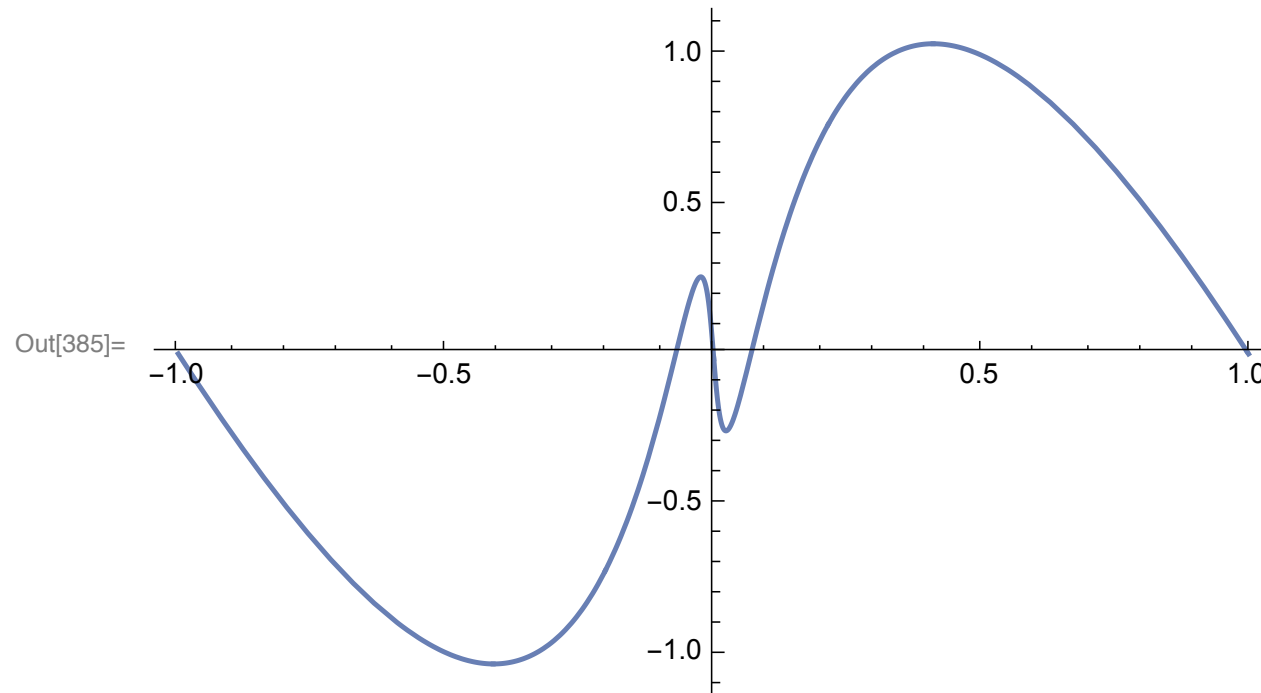
Two nodes, abnormal state, symmetric, $n = 1, k = 2$



$$B = 0.2, \alpha = 17.19$$

• Spectral function $g(z)$

Three nodes, abnormal state, antisymmetric, $n = 1, k = 3$



$$B = 10^{-4}, \alpha = 2.63$$

● Conclusion in the case $\mu = 0$

Abnormal states for massless exchange, as solutions of the BS equation, certainly exist.

However, minimal coupling constant
(for point-like particle!)

seems too large: $\alpha_{min} = \frac{\pi}{4} \Rightarrow Z = 107.$

What about abnormal states for finite range
(strong) interaction?

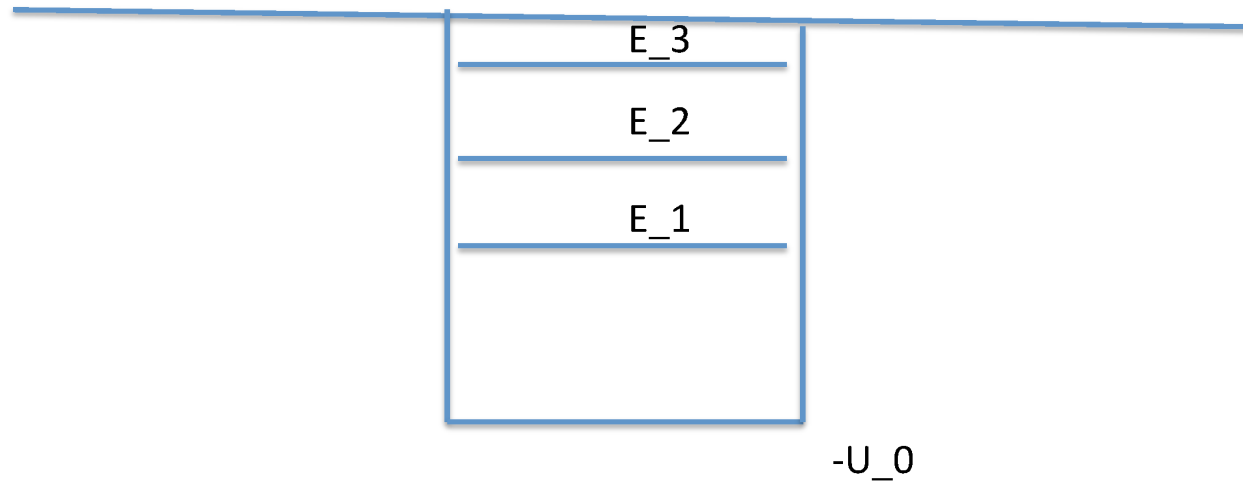
● Finite-range interaction, $\mu \neq 0$

- The number of states is normally finite.
- When the interaction decreases, the bound states disappear one after other:
 - first, the highest excited state disappears;
 - the next (lower) excited state disappears;
 - so on...
 - down to the ground state.

Any state disappears when finite-range interaction weakens.

There is its own critical coupling constant for each state.

● Example: potential well



Potential well: $U_{crit} = \frac{\pi^2}{8mr_0^2}$

For $\mu \neq 0$ the nice picture, when the abnormal spectrum exists when $\alpha > \alpha_c$,
this picture does not exist!

Do the abnormal states exist at all for $\mu \neq 0$?

To answer this question is our aim.

Deriving equation for $g(\gamma, z)$, $\mu \neq 0$

Reminder: Nakanishi representation

$$\Phi(k, p) = \int_{-1}^1 dz' \int_0^\infty d\gamma' \frac{g(\gamma', z')}{[\gamma' + m^2 - \frac{1}{4}M^2 - k^2 - p \cdot k z' - i\epsilon]^3}.$$

Massive case $\mu \neq 0$, $g(\gamma, z)$ depends on two variables γ, z .

Substitute $\Phi(k, p)$ in the BS equation and find equation for $g(\gamma, z)$.

- For a ladder kernel, in rather complicated form
(K. Kusaka, A.G. Williams, 1995)
- For arbitrary BS kernel, but in the form with two-integrals,
V.A. Karmanov and J. Carbonell, 2006:

$$\int_0^\infty \frac{g(\gamma', z) d\gamma'}{\left[\gamma' + \gamma + z^2 m^2 + (1 - z^2) \kappa^2\right]^2} = \int_0^\infty d\gamma' \int_{-1}^1 dz' V(\gamma, z; \gamma', z') g(\gamma', z')$$

- For the ladder kernel, in the canonical form (one integral in r.h..s.),
T. Frederico, G. Salmè and M. Viviani, 2014
- For arbitrary BS kernel, in the canonical form,
J. Carbonell, T. Frederico, V.A. Karmanov, 2017

Canonical form: $g(\gamma, z) = \int_0^\infty d\gamma' \int_{-1}^1 dz' N(\gamma, z; \gamma', z') g(\gamma', z')$

Equation for $g(\gamma, z)$ was analyzed analytically and solved numerically. More solutions (symmetric and antisymmetric) were found than in the corresponding Schrödinger equation. Some of them survive in the non-relativistic limit (and turn into the solutions of the Schrödinger equation), some of them (the abnormal ones) disappear in the non-relativistic limit.

The abnormal states for massive exchange $\mu \neq 0$ exist!
(We have found them!)

● Non-relativistic limit

Relativity exists since the speed of light c is finite and **the same** in any frame.

Non-relativistic limit is $c \rightarrow \infty$.

We should restore c in the BS equation and take the limit $c \rightarrow \infty$ (analytically and/or numerically).

Restoring c :

$$m \rightarrow mc^2, \quad M \rightarrow Mc^2, \quad \alpha = \frac{e^2}{\hbar c} \rightarrow \frac{\alpha}{c},$$

G. Wanders, *Limite non-relativiste d'une équation de Bethe-Salpeter*,

Helvetica Physica Acta, 1957 (in French).

Derivation of Schrödinger equation from the BS one (in the limit $c \rightarrow \infty$)

(Equal time) wave function:

$$\psi(\vec{p}) = \int_{-\infty}^{\infty} \Phi(\vec{p}, p_0) dp_0 = \int_0^{\infty} \frac{g(\gamma, 0) d\gamma}{(\gamma + \vec{p}^2 + \kappa^2)}$$

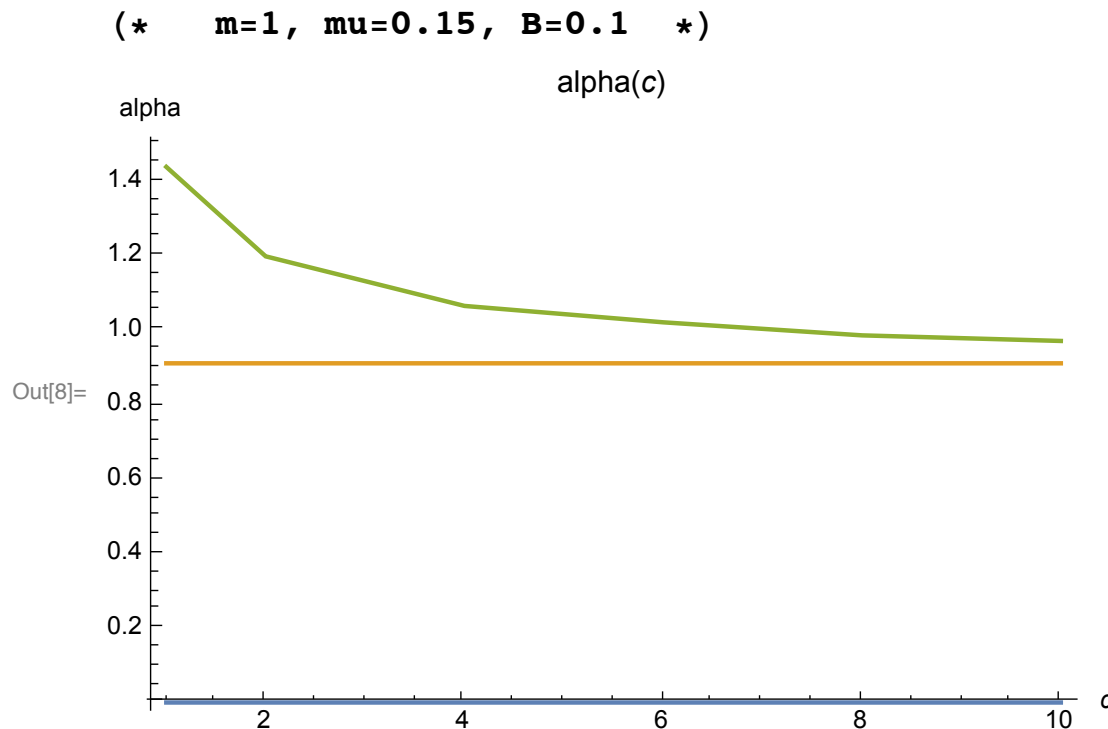
Since equation for g is known, one can derive equation for $\psi(\vec{p})$ (normal state):

$$\psi(\vec{p}) = \frac{4m\pi\alpha}{(\vec{p}^2 + \kappa^2)} \int \frac{d^3q}{(2\pi)^3} \frac{1}{((\vec{p} - \vec{q})^2 + \mu^2)} \psi(\vec{q})$$

$$\Rightarrow \left(\frac{\hat{\vec{p}}^2}{m} + V(r) \right) \psi(\vec{r}) = E\psi(\vec{r}), \quad V(r) = -\frac{\alpha}{r} e^{-\mu r}$$

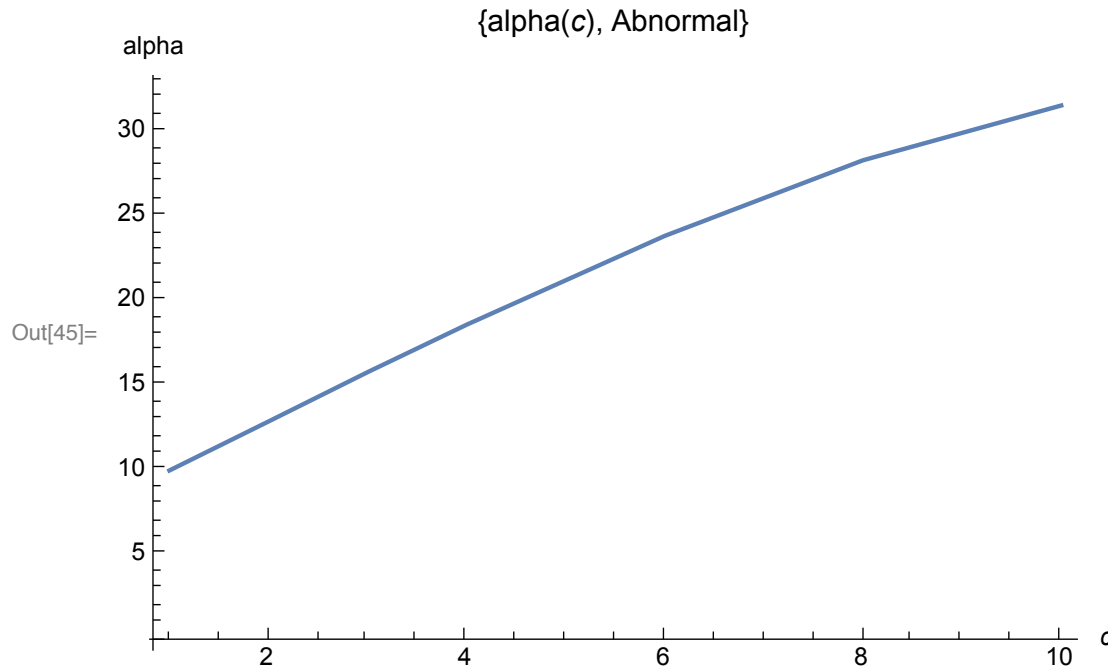
• Dependence $\alpha(c)$ for a normal solution

We repeat the calculations for a set of values of speed of light $1 \leq c \leq 10$ and find the dependence $\alpha(c)$.



Dependence of the coupling constant α
(for a normal state, $\mu = 0.15$, $B = 0.1$) on speed of light c

• Dependence $\alpha(c)$ for an abnormal state



Dependence of the coupling constant α

(for an abnormal state, $\mu = 0.15$, $B = 0.1$) on speed of light c

For normal state: $\alpha(c \rightarrow \infty) \rightarrow$ finite (nonrelativistic) limit.

For abnormal state: $\alpha(c \rightarrow \infty)$ increases without any limit.

• Question

If the "abnormal" states exist for a finite-range interaction, how to distinguish them from the "normal" ones?

In other words:

solving BS equation, we find an energy level.

What is its nature: normal or abnormal?

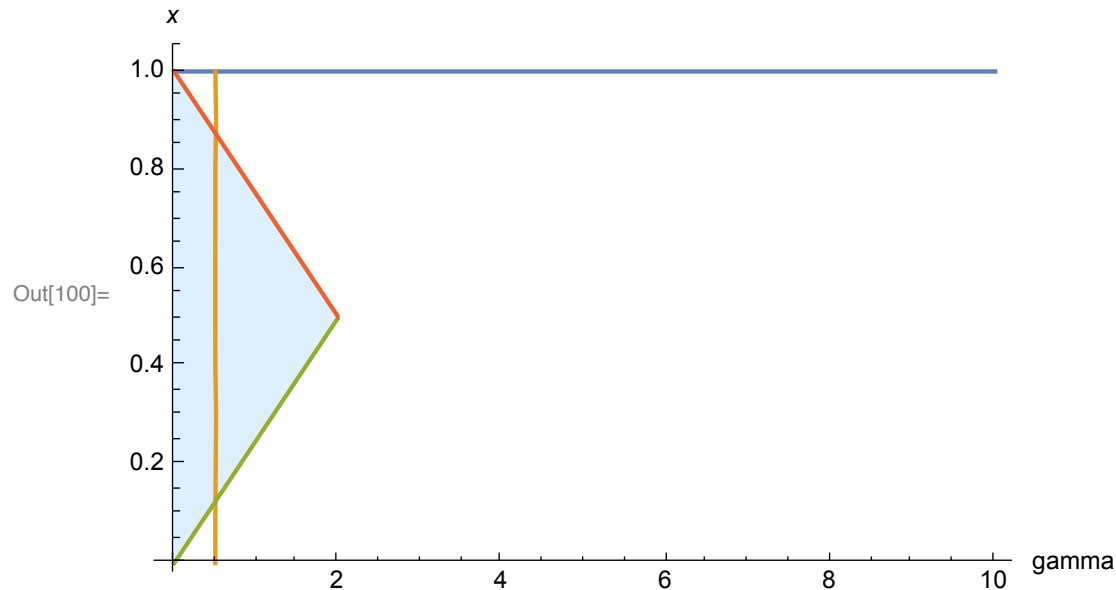
Introduce new variable x :

$$x = \frac{1}{2}(1 + z), \quad -1 \leq z \leq 1 \Rightarrow 0 \leq x \leq 1$$

$$\psi(\gamma, z) \Rightarrow \psi(\gamma, x)$$

• Domain of $g(\gamma, x)$

(* Domain: $0 < \text{gamma} < \text{infinity}, 0 < x < 1$ *)

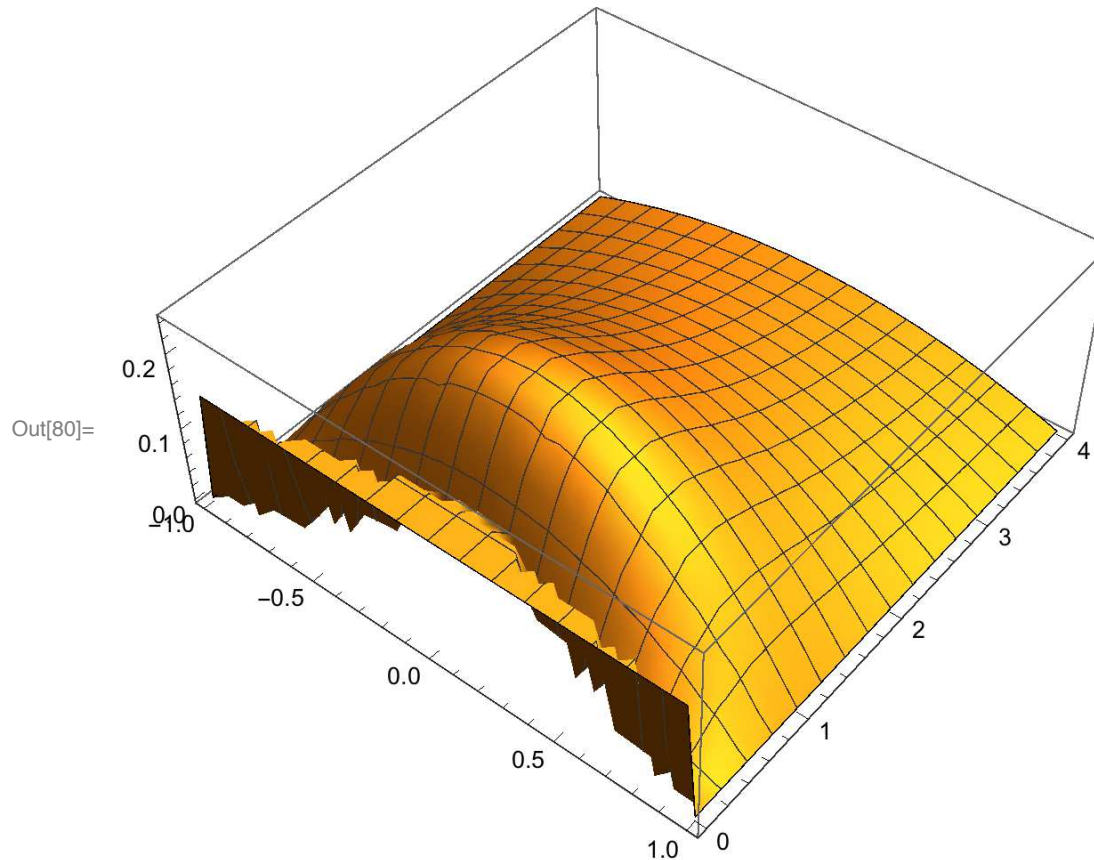


Domain $0 \leq x \leq 1$, $0 \leq \gamma, < \infty$ where $g(\gamma, x)$ is defined.

Light blue is domain where $g(\gamma, x) = \text{const}$:

$$\gamma \leq \gamma_0(x) \sim \begin{cases} \mu x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ \mu(1 - x), & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

• 3D plot of $g(\gamma, z)$



Weight function $g(\gamma, z)$

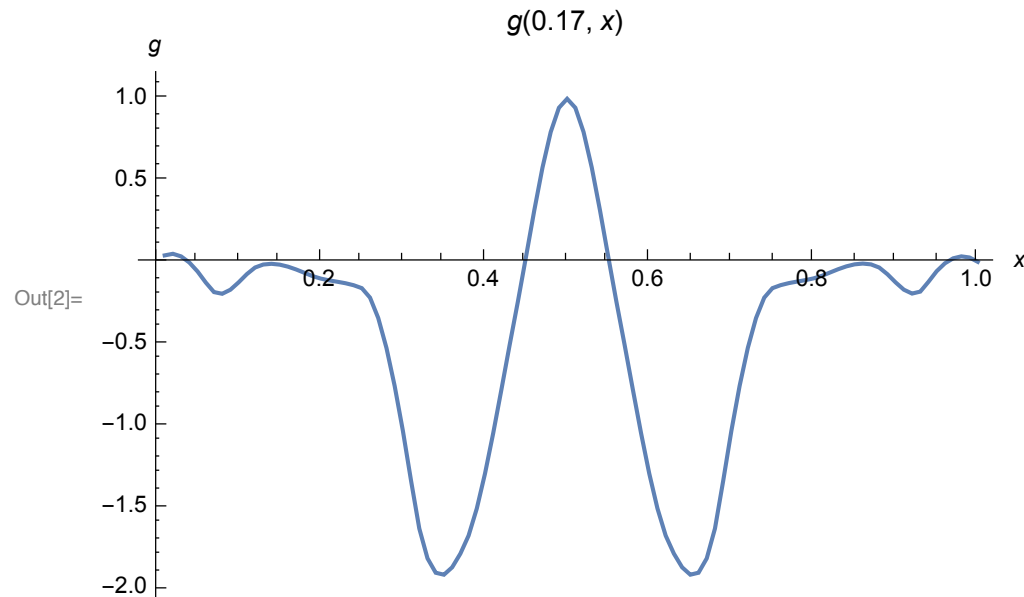
If $\gamma < \gamma_0 \approx (1 - |z|)2\mu\sqrt{mB}$, then $g(\gamma, z) = \text{const}$
(vs. γ and vs. z).

If $g(\gamma, z)$ is antisymmetric, then $\text{const} = 0$.

• Numerical solution $g(\gamma, x)$ ($c = 1$)

$$x = \frac{1}{2}(1 + z), \quad -1 \leq z \leq 1 \Rightarrow 0 \leq x \leq 1$$

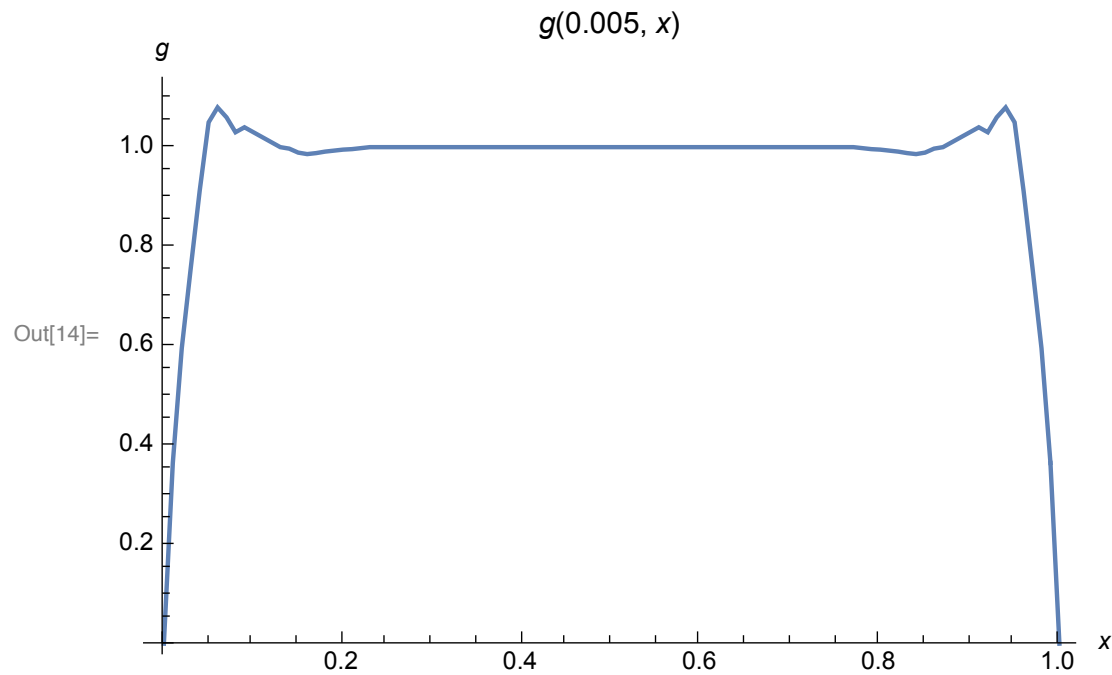
(* alpha=1.4375 Normalized g(gamma=0.17,x=0.5)=1 *)



Weight function $g(\gamma = 0.17, x)$

- $g(\gamma = 0.005, x)$

Normal state

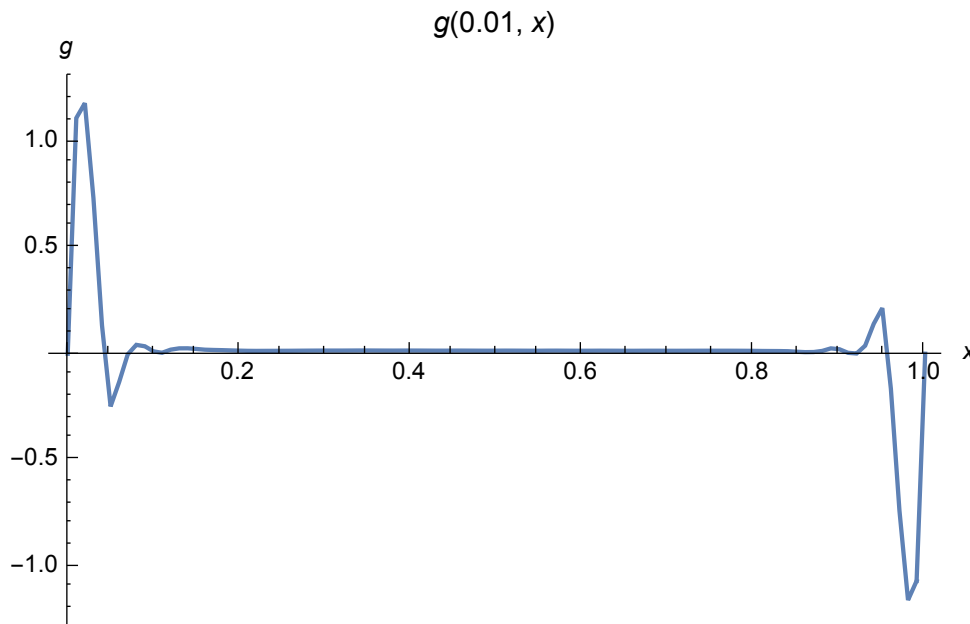


Weight function $g(\gamma = 0.005, x)$

- $g(\gamma = 0.005, x)$

Antisymmetric abnormal state

(* alpha=9.8882 Normalized $g(\gamma=0.01, x=0.025)=1$ *)

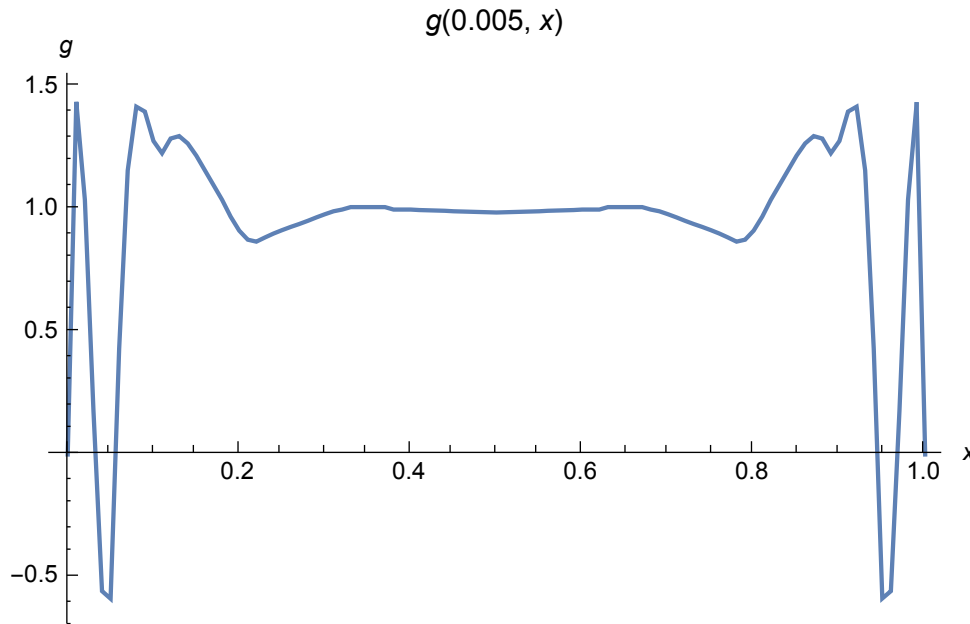


Weight function $g(\gamma = 0.005, x)$

- $g(\gamma = 0.005, x)$

Symmetric abnormal state

(* alpha=11.2859 Normalized $g(\gamma=0.005, x=0.4)=1$ *)



Weight function $g(\gamma = 0.005, x)$

● Criterion

$g(\gamma < \gamma_0(x), x)$ is always constant
(both for normal and abnormal states).

Outside, $g(\gamma, x) \rightarrow 0$ when $x \rightarrow 0, 1$.

For normal states:

in vicinity of $x = 0, 1$, it has no nodes.

For abnormal states:

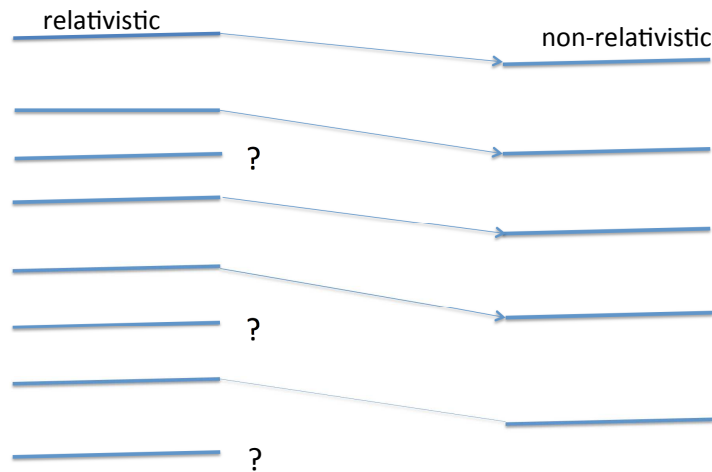
in vicinity of $x = 0, 1$, it has nodes.

**In this way, one can distinguish abnormal states
from the normal ones,**

like in the massless case, but in the domain $g(\gamma < \gamma_0(x), x \approx 0, 1)$.

This is a phenomenological observation,
based on numerical calculations
(never any exclusions were found!),
but not yet a theorem.

● Non-relativistic vs. relativistic states



There is no any theorem proving one-to-one correspondence of the energy levels in relativistic and non-relativistic equations.

Therefore, one can expect that in a relativistic system there exist the states which have no non-relativistic counterpart.

If so, it is senseless to analyze them in the Schrödinger equation framework.

● Conclusions

- Like Dirac equation predicts antiparticles, the BS equation predicts the states having **pure relativistic origin** (not given by the Schrödinger equation).
- We know how to distinguish them from the normal ones (by behavior of solutions $g(\gamma, x)$ vs. x at small γ).
 - This is important for understanding the relativistic few-body systems
- It is interesting to analyse, from this point of view, the spectra of particles.

● To be calculated

- Elastic electromagnetic formfactor
- Transition e.m. formfactor (normal \leftrightarrow abnormal states).

BS amplitude: $\Phi(x_1, x_2; p) = \langle 0|T\{\phi(x_1)\phi(x_2)\}|p\rangle$.

$\Phi(x_1, x_2; p)$ is "two-body" (depends on two variables).

Whereas, $|p\rangle$ is many-body (it contains not only two-body but also higher Fock states).

- To calculate the two-body and many-body contributions in $|p\rangle$.

Thank you!