# UNPHYSICAL ENERGY SHEETS AND RESONANCES IN THE FRIEDRICHS-FADDEEV MODEL


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# History and problem setup

In 1938, Kurt Friedrichs[1] considered a model Hamiltonian of the form

$$H_\varepsilon = H_0 + \varepsilon V$$

with $H_0$, the multiplication by the independent variable $\lambda$ (a kind of the position operator),

$$(H_0 f)(\lambda) = \lambda f(\lambda), \qquad \lambda \in (-1,1) \subset \mathbb{R}, \qquad f \in L_2(-1,1),$$

and $V$, an integral operator,

$$(Vf)(\lambda) = \int_a^b V(\lambda,\mu) f(\mu) d\mu,$$

where the kernel $V(\lambda,\mu)$ is a continuous function in $\lambda,\mu \in [a,b]$ of a Hölder class. Furthermore, he assumed that

$$V(-1,\mu) = V(1,\mu) = V(\lambda,-1) = V(\lambda,1) \quad \text{for any } \lambda,\mu \in [0,1].$$

Hermitian (self-adjoint) operator $H_0$ has (absolutely) continuous spectrum that fills the segment $[-1,1]$. Friedrichs studied what happens to the continuous spectrum of $H_0$ under the perturbation $\varepsilon V$.

[1] K.Friedrichs, *Über die Spectralzerlegung eines Integraloperators*, Math. Ann. **115** (1938), 249–272.

Friedrichs succeeded to prove that if $\varepsilon$ is sufficiently small then $H_\varepsilon$ and $H_0$ are similar, which means that the spectrum of $H_\varepsilon$ is also continuous and fills $[-1,1]$.

In a 1948 paper[2], Friedrichs has extended this result to the case where the unperturbed Hamiltonian $H_0$ is the multiplication by independent variable in the Hilbert space

$$\mathfrak{H} = L_2(\Delta, \mathfrak{h})$$

of square-integrable vector-valued functions

$$f : \Delta \to \mathfrak{h}, \qquad \|f\|^2 = \int_\Delta d\lambda \, \|f(\lambda)\|_{\mathfrak{h}}^2,$$

where $\Delta$ is a finite or infinite interval on the real axis,

$$\Delta = (a,b), \quad \text{with } -\infty \leq a < b \leq +\infty$$

and $\mathfrak{h}$ is an auxiliary Hilbert space (finite- or infinite-dimensional). In this case, it is assumed that for every $\lambda, \mu \in \Delta$ the quantity $V(\lambda,\mu)$ is a bounded linear operator on $\mathfrak{h}$, that $V(\lambda,\mu) = V(\lambda,\mu)^*$, and that $V$ is a Hölder continuous operator-valued function of $\lambda, \mu$. This time, Friedrichs proves that, under certain additional assumptions on $V(\lambda,\mu)$, for sufficiently small $\varepsilon$ the perturbed

[2] K.O.Friedrichs, *On the perturbation of continuous spectra*, Comm. Pure Appl. Math. **1** (1948), 361-406.

operator $H_\varepsilon = H_0 + \varepsilon V$ is unitarily equivalent to the unperturbed one, $H_0$, and thus the spectrum of $H_\varepsilon$ is absolutely continuous and fills the interval $\Delta$.

In 1958, O.A. Ladyzhenskaya and L.D. Faddeev[3] have dropped the smallness requirement on $V$ completely and considered the model operator

$$\begin{gathered} H = H_0 + V, \\ (H_0 f)(\lambda) = \lambda f(\lambda), \quad (Vf)(\lambda) = \int_\Delta V(\lambda,\mu) f(\mu) d\mu, \\ f \in L_2(\Delta, \mathfrak{h}), \quad \Delta = (a,b), \end{gathered} \tag{1}$$

with NO small $\varepsilon$ in front of $V$. Instead, they require compactness of the value of $V(\lambda,\mu)$ as an operator in $\mathfrak{h}$ for any $\lambda,\mu \in [a,b]$.

Proofs (and an extension) are given in a Faddeev's 1964 work[4]: Complete version of the scattering theory for the model under consideration. (The paper of 1964 may be viewed as a relatively simple introduction to the methods and ideas he used in his celebrated analysis of the three-body problem.)

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[3] O.A. Ladyzhenskaya and L.D. Faddeev, *On continuous spectrum perturbation theory*, Dokl. Akad. Nauk SSSR **120** (1958), 1187–1190.

[4] L.D.Faddeev, *On a model of Friedrichs in the theory of perturbations of the continuous spectrum*, Trudy Mat. Inst. Steklov. **73** (1964), 292–313.

Faddeev's detail study of the Hamiltonian (1) is the first reason why this Hamiltonian is often called the Friedrichs-Faddeev model. The second reason is related to the fact that the 1948 Friedrichs' paper contains another important ($2 \times 2$ block matrix) operator model that is called "simply" Friedrichs' model. The second model works for Feshbach resonances.

Many people used or worked on the Friedrichs/Friedrichs-Faddeev models and their generalizations (Albeverio, Lakaev, Gadella, Pavlov, Pronko, Isozaki, Richard,...). Source of explicitly solvable examples.

Notice that the typical two-body Schrödingrer operator may be viewed as a particular case of the Friedrichs-Faddeev model with $a = 0$ and $b = +\infty$. Simply consider the c.m. Schrödingrer operator in the momentum $(\boldsymbol{k})$ space and make the variable change $|\boldsymbol{k}|^2 \to \lambda$; in this case the internal space is $\mathfrak{h} = L_2(S^2)$, i.e. the space of square-integrable functions on the unit sphere in $\mathbb{R}^3$).

It turned out that there is a gap in study of analytical properties and structure of the FF $T$- and $S$-matrices on uphysical sheets of the energy plane. We fill this gap by using the ideas and approach from a couple of the speaker's works[5],[6].

Furthermore, we will perform a *complex deformation* (a generalization of complex scaling) of the FF Hamiltonian. Discrete spectrum of the complexly deformed Hamiltonian contains the "complex scaling resonances". We show these resonances are simultaneously the scattering matrix resonances.

With Friedrichs-Faddeev model — downgrading / great simplification of the problem in both cases...

[5]A. K. Motovilov, *Analytic continuation of S matrix in multichannel problems*, Theor. Math. Phys. **95** (1993), 692–699.
[6]A. K. Motovilov, *Representations for the three–body T–matrix, scattering matrices and resolvent in unphysical energy sheets*, Math. Nachr. **187** (1997), 147–210.

# Structure of the *T*- and *S*-matrices for the FF model on unphysical energy sheets

First, let us recollect the description of the Friedrichs-Faddeev (FF) model. We assume that $\mathfrak{h}$ is an auxiliary ("internal") Hilbert space and $\Delta=(a,b)$, an interval on $\mathbb{R}$,

$$-\infty\leq a<b\leq+\infty.$$

Hilbert space of the problem is the space of square-integrable $\mathfrak{h}$-valued functions on $\Delta$,

$$\mathfrak{H}=L_2(\Delta,\mathfrak{h})\quad\text{(consists of functions } f:\Delta\to\mathfrak{h}\text{)},$$

with scalar product

$$\langle f,g\rangle=\int_a^b d\lambda\,\langle f(\lambda),g(\lambda)\rangle_{\mathfrak{h}}.$$

Surely, the norm on $\mathfrak{H}$ is given by

$$\|f\|=\left(\int_a^b d\lambda\,\|f(\lambda)\|_{\mathfrak{h}}^2\right)^{1/2}.$$

$\langle\cdot,\cdot\rangle_{\mathfrak{h}}$ and $\|\cdot\|_{\mathfrak{h}}$ denote, respectively, the scalar product and norm on $\mathfrak{h}$.

Unperturbed Hamiltonian $H_0$ is the operator of multiplication by $\lambda$

$$(H_0 f)(\lambda) = \lambda f(\lambda), \qquad \lambda \in \Delta, \quad f \in \mathfrak{H} = L_2(\Delta, \mathfrak{h}),$$

The perturbation (potential) $V$ is given by the integral

$$(Vf)(\lambda) = \int_a^b V(\lambda,\mu) f(\mu) d\mu,$$

where for each $\lambda,\mu \in (a,b)$ the value of $V(\lambda,\mu)$ is a compact operator in $\mathfrak{h}$. We assume $V(\lambda,\mu)$ admits analytic continuation both in $\lambda$ and $\mu$ into some domain $\Omega \subset \mathbb{C}$ containing $\Delta$ (that is, we assume

$$V(\lambda,\mu) \quad \text{is holomorphic in both} \quad \lambda,\mu \in \Omega, \quad (a,b) \subset \Omega).$$

Furthermore, $V(\lambda,\mu) = V(\mu,\lambda)^*$ for real $\lambda,\mu \in \Delta$ (for Hermiticity of $V$). In addition: $V(a,\mu) = V(b,\mu) = V(\lambda,a) = V(\lambda,b) = 0$ or suitable requirements on the rate of decreasing of $V(\lambda,\mu)$ as $|\lambda|,|\mu| \to \infty$ (in case of infinite $a$ or/and $b$).

As usually, the total Hamiltonian is

$$H = H_0 + V.$$

Also we use the standard notation: for $z$ outside the corresponding spectrum,

$$R_0(z) = (H_0 - z)^{-1}, \quad R(z) = (H - z)^{-1}, \quad T(z) = V - VR(z)V.$$

(The kernel $T(\lambda, \mu, z)$ is a $\mathscr{B}(\mathfrak{h})$-valued function of $\lambda, \mu, z$.)

Recall that (for admissible $z$, in particular for $z \notin \mathrm{spec}(H_0) \cup \mathrm{spec}(H)$)

$$R(z) = R_0(z) - R_0(z)T(z)R(z). \tag{2}$$

Thus, the spectral problem for $H$ is reduced to the study of the $T$-matrix $T(z)$.

From Faddeev (1964): $T(\lambda, \mu, z)$ is well-behaved function of $\lambda, \mu \in \Delta$ and $z$ on the complex plane $\mathbb{C}$ punctured at $\sigma_p(H)$ and cut along $[a, b]$. $T(\lambda, \mu, z)$ has limits

$$T(\lambda, \mu, E \pm i0), \quad E \in \Delta \setminus \sigma_p(H)$$

that are (in our case) smooth in $\lambda, \mu \in \Delta$. The scattering matrix for the pair $(H_0, H)$ is given by

$$S_+(E) = I_{\mathfrak{h}} - 2\pi \mathrm{i}\, T(E, E, E + \mathrm{i}0), \quad E \in (a, b) \setminus \sigma_p(H).$$

Notice that the eigenvalue set $\sigma_p(H)$ is finite.

Take a look of the Lippmann-Schwinger equations

$$T(z) = V - VR_0(z)T(z) \quad \text{and} \quad T(z) = V - T(z)R_0(z)V$$

for the $T$-matrix $T(z)$:

$$T(\lambda,\mu,z) = V(\lambda,\mu) - \int_a^b d\nu \quad \frac{V(\lambda,\nu)T(\nu,\mu,z)}{\nu - z}, \tag{3}$$

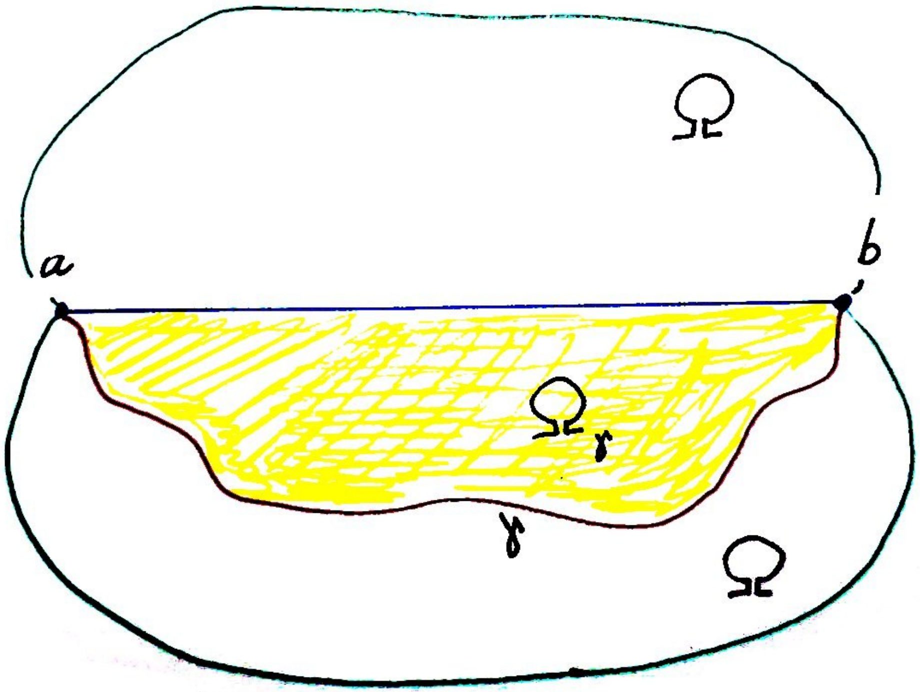
$$T(\lambda,\mu,z) = V(\lambda,\mu) - \int_a^b d\nu \quad \frac{T(\lambda,\nu,z)V(\nu,\mu)}{\nu - z}, \tag{4}$$

$$z \notin (a,b), \quad \lambda,\mu \in (a,b)$$

Clearly, (3) and (4) imply analyticity of $T(\lambda,\mu,z)$ in $\lambda \subset \Omega$ and in $\mu \subset \Omega$, respectively.

**Proposition 1.** One can replace $(a,b)$ in (3) and (4) by arbitrary piecewise smooth Jordan contour $\gamma \subset \Omega$ obtained by continuous deformation from $(a,b)$ provided that the end points are fixed and the point $z$ during the transformation is avoided.

In the following $\mathbb{C}^+ = \{z \in \mathbb{C} \,|\, \mathrm{Im}\, z > 0\}$ $\quad$ ($\mathbb{C}^- = \{z \in \mathbb{C} \,|\, \mathrm{Im}\, z < 0\}$) denotes the upper (lower) halfplane of $\mathbb{C}$.



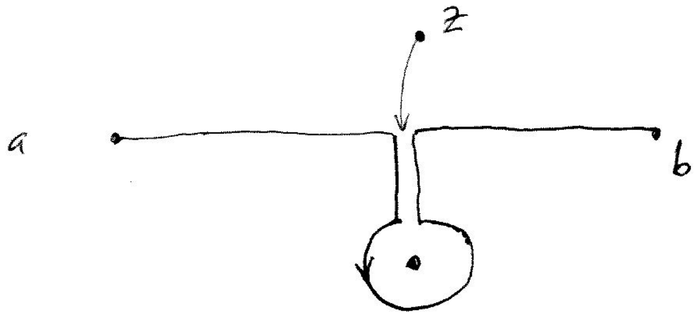

In particular, Proposition 1 implies that for $\gamma \subset \Omega \cap \mathbb{C}^{\pm}$ one can equivalently write

$$T(\lambda,\mu,z) = V(\lambda,\mu) - \int_{\gamma} d\nu \, \frac{V(\lambda,\nu)T(\nu,\mu,z)}{\nu - z}, \quad \lambda,\mu \in \Omega, \quad z \in \mathbb{C} \setminus \Omega_{\gamma}, \tag{5}$$

where the set $\Omega_\gamma$ in $\mathbb{C}$ confined by (and containing) the interval $[a,b]$ and the curve $\gamma$.

In principle, one could allow $z$ to enter $\Omega_\gamma$ from above and then solve (or at least prove the solvability of) the equation (5). In the following, if one tries to re-establish the original integration over the interval $(a,b)$, it will be necessary to take the residue at the pole $z$. That is, the Lippmann-Schwinger equation (5) changes its form and, hence, for $z \in \Omega \cup \mathbb{C}^-$ the solution $T'(\lambda,\mu,z)$ is taken, in fact, on the part $\Omega \cup \mathbb{C}^-$ belonging to the unphysical sheet of the Riemann energy surface of $T$.

In fact we can solve the continued equation explicitly!



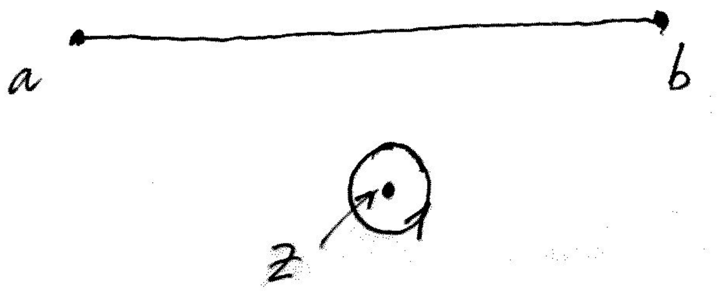

Simply start with $\gamma = (a,b)$ and $\operatorname{Im} z > 0$:

$$T(\lambda,\mu,z) = V(\lambda,\mu) - \int_a^b d\nu \quad \frac{V(\lambda,\nu)T(\nu,\mu,z)}{\nu - z}, \qquad (6)$$

$$\lambda,\mu \in \Omega.$$

↖ Intermediate contour $\gamma$.

↙ Final contour $\gamma$.




After the transformation of the contour, pulling $z$ downstairs, and computing the residue at $\nu = z$:

$$T'(\lambda,\mu,z) = V(\lambda,\mu) - 2\pi \mathrm{i}\, V(\lambda,z)T'(z,\mu,z) - \int_a^b d\nu \quad \frac{V(\lambda,\nu)T'(\nu,\mu,z)}{\nu - z}, \qquad (7)$$

$$\lambda,\mu \in \Omega, \quad z \in \Omega \cap \mathbb{C}^-.$$

"Prime" in $T'$ means that the $T$ is already taken on the unphysical sheet $\Pi_-$ sticked to the physical energy sheet along the upper rim of the cut $(a,b)$.

$$\boxed{T'(\lambda,\mu,z)+\int_a^b d\nu \quad \frac{V(\lambda,\nu)T'(\nu,\mu,z)}{\nu-z}=V(\lambda,\mu)-2\pi\mathrm{i}V(\lambda,z)T'(z,\mu,z),} \tag{8}$$

$$\lambda,\mu\in\Omega,\quad z\in\Omega\cap\mathbb{C}^-.$$

$T'(\lambda,\mu,z)$ is an “off-shell” object
$T'(z,\mu,z)$ is “half-on-shell” (with respect to the first argument)

Equation (8) allows us to express the off-shell $T'$ exclusively through the half-on-shell $T'$ by taking into account that, on the physical sheet,

$$(I+VR_0(z))T(z)=V \quad\Longrightarrow\quad (I+VR_0(z))^{-1}V=T(z), \qquad z\notin\sigma_p(H).$$

Thus, (8) implies

$$\boxed{T'(\lambda,\mu,z)=T(\lambda,\mu,z)-2\pi\mathrm{i}\,T(\lambda,z,z)T'(z,\mu,z).} \tag{9}$$

Next step: $T'(z,\mu,z)=T(z,\mu,z)-2\pi\mathrm{i}\,T(z,z,z)T'(z,\mu,z)$, which means

Tell on $z\notin\sigma(S_-(z))$

$S_-(z)T'(z,\mu,z)=T(z,\mu,z)$, that is, $\boxed{T'(z,\mu,z)=S_-(z)^{-1}T(z,\mu,z)}$

where the scattering matrix $S_-(z)$, $z\in\Omega\cap\mathbb{C}^-$ (on the physical sheet!), is given by

$$\boxed{S_-(z):=I_{\mathfrak{h}}+2\pi\mathrm{i}\,T(z,z,z).}\,.$$

Finally, from the relations obtained (write them once again),

$$T'(\lambda,\mu,z) = T(\lambda,\mu,z) - 2\pi\mathrm{i}\,T(\lambda,z,z)T'(z,\mu,z),$$
$$T'(z,\mu,z) = S_-(z)^{-1}T(z,\mu,z),$$

it follows that

$$\boxed{T'(\lambda,\mu,z) = T(\lambda,\mu,z) - 2\pi\mathrm{i}\,T(\lambda,z,z)S_-(z)^{-1}T(z,\mu,z).} \qquad (10)$$

All the entries on the r.h.s. part of (10) are taken on the physical sheet!

In a similar way we perform the continuation of $T(\lambda,\mu,z)$ from the lower half-plane $\mathbb{C}^-$ to the part $\Omega\cap\mathbb{C}^+$ of the unphysical energy sheet $\Pi_+$ attached to the physical sheet along the lower rim of the cut $(a,b)$.
Combined result (for both $\Pi_\ell$, $\ell=\pm 1$, identified with the respective sign $\pm$)

$$\boxed{T(\lambda,\mu,z)\big|_{z\in\Pi_\ell} = \left(T(\lambda,\mu,z) + 2\pi\mathrm{i}\,\ell\,T(\lambda,z,z)S_\ell(z)^{-1}T(z,\mu,z)\right)\big|_{z\in\mathbb{C}^\ell\cap\Omega}.}$$

R.h.s. entries are on the physical sheet,

$$S_\pm(z) = I_{\mathfrak{h}} \mp 2\pi\mathrm{i}\,T(z,z,z).$$

Whether $\Pi_-$ and $\Pi_+$ represent the same ("second") unphysical sheet, depends on the analytical properties of $V(\lambda,\mu)$ outside $\Omega$ (if available).

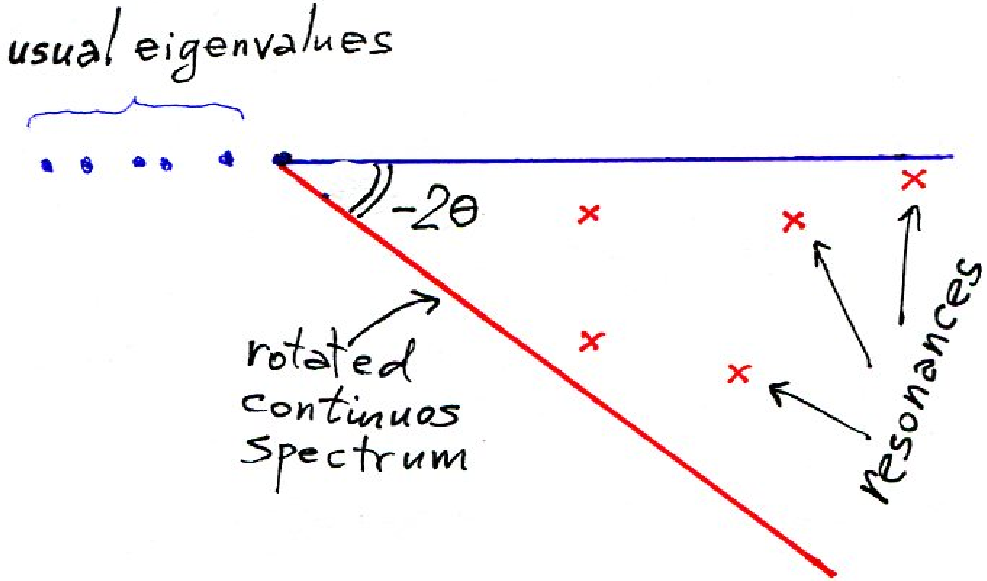
Continuation formulae for $T$ imply for the continuation of $S_{\pm}$ the following:

$$S_{\pm}(z)\big|_{\Pi_{\mp}} = S_{\mp}(z)^{-1}\big|_{z \in \mathbb{C}^{\mp} \cap \Omega}.$$

Thus, the resonances, e.g., on the unphysical sheet $\Pi_{-}$ are nothing but zeros of the operator-function $S_{-}(z) = I_{\mathfrak{h}} + 2\pi\mathrm{i}\, T(z,z,z)$ on the physical sheet. That is, the points $z \in \mathbb{C}^{-} \cap \Omega$ on the physical sheet where

$$S_{-}(z)\mathscr{A} = 0 \quad \text{for a non-zero vector } \mathscr{A} \in \mathfrak{h}.$$

# Friedrichs-Faddeev model and complex scaling




In the coordinate space, the standard complex scaling means the replacement of the original c.m. two-body Hamiltonian

$$H = -\Delta + \widehat{V}(\boldsymbol{r})$$

by the non-Hermitian operator

$$H(\theta) = -\mathrm{e}^{-2\mathrm{i}\theta}\Delta + \widehat{V}(\mathrm{e}^{\mathrm{i}\theta}\boldsymbol{r}),$$

for a non-negative $\theta \leq \pi/2$, provided the local potential $\widehat{V}(\boldsymbol{r})$ admits analytic continuation to a domain of complex $\mathbb{C}^3$-arguments $\boldsymbol{r}$.

Having performed the Fourier transform and then making the change $|\boldsymbol{k}|^2 \to \lambda$ one arrives at the complex version of the Friedrichs-Faddeev model

$$(H(\theta)f)(\lambda) = \mathrm{e}^{-2\mathrm{i}\theta}\lambda f(\lambda) + \mathrm{e}^{-2\mathrm{i}\theta}\int_0^\infty V(\mathrm{e}^{-2\mathrm{i}\theta}\lambda, \mathrm{e}^{-2\mathrm{i}\theta}\mu) f(\mu)d\mu, \qquad (11)$$
$$f \in L_2\big(\mathbb{R}^+, L_2(S^2)\big).$$

The operator-valued function $V(\lambda,\mu)$ $(= V(\lambda-\mu)$ in the case of local $\widehat{V}$) is explicitly expressed through the Fourier transform of $\widehat{V}$. For every admissible $\lambda,\mu \in \mathbb{C}$ the value of $V(\lambda,\mu)$ is a (compact) operator in $\mathfrak{h} = L_2(S^2)$.

The Hamiltonian (11) may be immediately rewritten as the Friedrichs-Faddeev model on a contour in the complex plane,

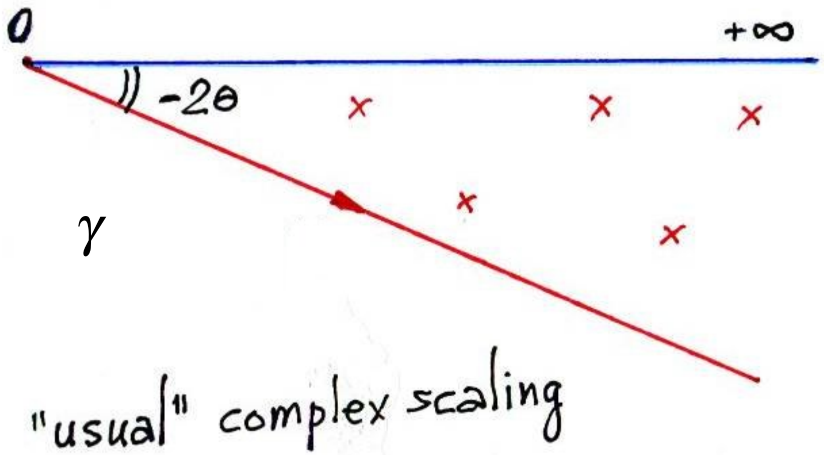
$$(H_\gamma f)(\lambda) = \lambda f(\lambda) + \int_\gamma V(\lambda,\mu) f(\mu) d\mu, \qquad \lambda \in \gamma,$$

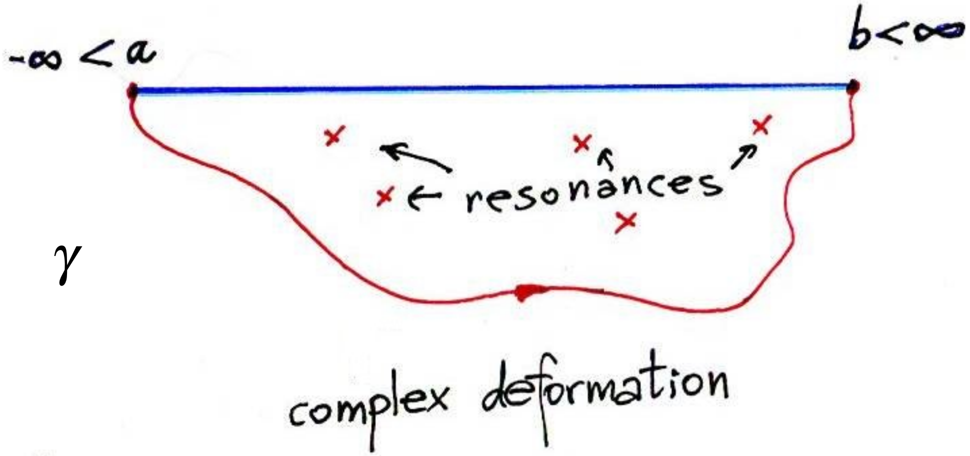
where

$$\gamma = \mathrm{e}^{-2\mathrm{i}\theta}\mathbb{R}^+ := \{z \in \mathbb{C} \,|\, z = \mathrm{e}^{-2\mathrm{i}\theta}x,\, 0 \le x < \infty\}$$

and $f \in L_2\big(\gamma, L_2(S^2)\big)$.

In the two-body problem case, we assume that $V(\lambda,\mu)$ is analytic in both $\lambda$ and $\mu$ on some domain $\Omega\subset\mathbb{C}$ containing the positive semiaxis $\mathbb{R}^+$ and symmetric with respect to $\mathbb{R}^+$. In addition, $\|V(\lambda,\mu)\|$ should decrease sufficiently rapidly as $|\lambda|\to\infty$ and/or $|\mu|\to\infty$ (in order to ensure compactness of the arising integral operators).



In the following we consider a family of the Friedrichs-Faddeev Hamiltonians

$$H_\gamma = H_{0,\gamma} + V_\gamma$$

associated with Jordan curves $\gamma \subset \Omega$ originating in $(a,b)$. Here $\Omega$ denotes the holomorphy domain of $V(\lambda,\mu)$ in $\lambda$ and $\mu$; $\Omega$ may not include $a$ and/or $b$;

$$\big(H_{0,\gamma}f\big)(\lambda) = \lambda f(\lambda) \quad \text{and} \quad \big(V_\gamma f\big)(\lambda) = \int_\gamma V(\lambda,\mu) f(\mu) d\mu, \quad \lambda \in \gamma,$$

where $f \in L_2(\gamma,\mathfrak{h})$,

$$L_2(\gamma,\mathfrak{h}) = \left\{ f : \gamma \to \mathfrak{h} \;\middle|\; \int_\gamma |d\lambda| \, \|f(\lambda)\|^2_{\mathfrak{h}} < \infty \right\}.$$

## Equivalence of the complex rotation resonances and scattering resonances in the Friedrichs-Faddeev model

From now on, for simplicity, we assume that both $a$ and $b$ are finite and, in addition, $V(\lambda,\mu)$ is continuous at $\lambda = a,b$ and $\mu = a,b$; $a,b \in \partial\Omega$.

As usually, we introduce the $T$-matrices for the pairs $(H_{0,\gamma}, H_\gamma)$,

$$T_\gamma(z) = V_\gamma - V_\gamma(H_\gamma - z)^{-1}V_\gamma, \quad z \notin \sigma(H_\gamma). \tag{12}$$

For $R_\gamma(z) = (H_\gamma - z)^{-1}$ we have

$$R_\gamma(z) = R_{0,\gamma}(z) - R_{0,\gamma}(z)T_\gamma(z)R_{0,\gamma}(z),$$

where $R_{0,\gamma}(z) = (H_{0,\gamma} - z)^{-1}$, $z \notin \sigma(H_{0,\gamma})$.

Notice that $H_{0,\gamma}$ has only continuous spectrum and this spectrum coincides with the curve $\gamma$. Thus, the discrete eigenvalues of $H_\gamma$ are nothing but the poles of the operator-valued function $T_\gamma(z)$.

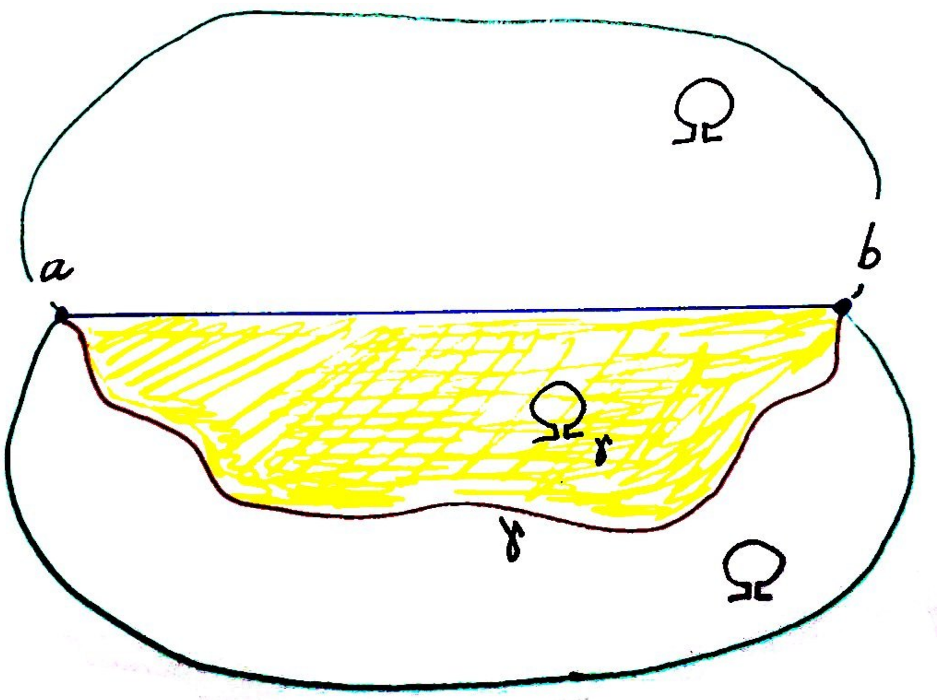
Already from (12) one may conclude that, for any fixed $z \notin \sigma(H_\gamma)$, the kernel $T_\gamma(\lambda, \mu, z)$ is holomorphic in the variables $\lambda, \mu \in \Omega$ (since $V$ is holomorphic). Indeed, (12) means

$$T_\gamma(\lambda, \mu, z) = V(\lambda, \mu) + \int_\gamma d\mu' \int_\gamma d\lambda' \, V(\lambda, \mu')R_\gamma(\mu', \lambda', z)V(\lambda', \mu).$$

One may pull $\lambda$ and $\mu$ anywhere in $\Omega$. And this will be true after analytic continuation of $R_\gamma(\mu', \lambda', z)$ in $z$ through $\gamma$!

Now look at the Lippmann-Schwinger equation for $T_\gamma$,

$$T_\gamma(\lambda,\mu,z) = V(\lambda,\mu) - \int_\gamma d\nu \quad \frac{V(\lambda,\nu)T_\gamma(\nu,\mu,z)}{\nu - z}, \quad z \notin \overline{\gamma}, \quad \lambda,\mu \in \gamma. \tag{13}$$




Let $z$ lie outside the set $\Omega_\gamma$ in $\mathbb{C}$ confined by (and containing) the interval $[a,b]$ and the curve $\gamma$. Consider for such a $z$ the Lippmann-Shwinger equation for the "original" $T$-matrix — it is associated with the interval $(a,b)$:

$$T(\lambda,\mu,z) = V(\lambda,\mu) - \int_a^b d\nu \quad \frac{V(\lambda,\nu)T(\nu,\mu,z)}{\nu - z}, \quad z \notin \Omega_\gamma, \quad \lambda,\mu \in (a,b). \tag{14}$$

Since both $V(\lambda,\cdot,z)$ and $T(\lambda,\cdot,z)$ for fixed $z \notin \Omega_\gamma(\cup\sigma_d(H))$ are holomorphic in $\lambda \in \Omega$, one may transform the interval $[a,b]$ into the contour $\gamma$ and obtain:

$$T(\lambda,\mu,z) = V(\lambda,\mu) - \int_\gamma d\nu \quad \frac{V(\lambda,\nu)T(\nu,\mu,z)}{\nu - z}, \quad z \notin \Omega_\gamma, \quad \lambda,\mu \in (a,b). \tag{15}$$

Compare (13) and (15). Pull $\lambda,\mu$ on $\gamma$. Uniqueness theorem for the solution to (15) implies:

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$T_\gamma(\lambda,\mu,z) = T(\lambda,\mu,z)$ whenever $\lambda,\mu \in \gamma$, $z \in \Omega \setminus \Omega_\gamma$ (and $z \notin \sigma_d(H)$).

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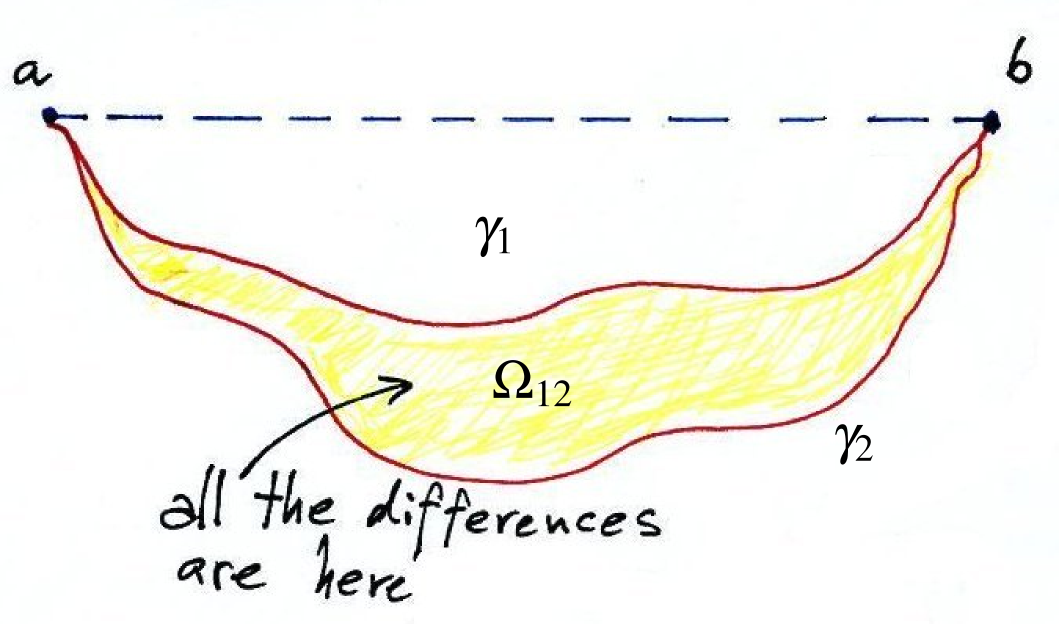


Similarly,

$$\boxed{T_{\gamma_1}(\lambda,\mu,z) = T_{\gamma_2}(\lambda,\mu,z)}$$

whenever $\lambda,\mu \in \gamma_1$ or $\lambda,\mu \in \gamma_2$, $z \in \Omega \setminus \Omega_{12}$ (and $z \notin \sigma_d(H)$).

Finally, by the uniqueness principle for analytic continuation, for $z$ inside $\Omega_\gamma$ the kernel $T_\gamma(\lambda,\mu,z)$ represents just the analytic continuation of $T(\lambda,\mu,\cdot)$ to the interior of $\Omega_\gamma$ lying in the unphysical sheet. Hence, the poles of $T_\gamma(z)$ within $\Omega_\gamma$ represent resonances of the original Friedrichs-Faddeev Hamiltonian on $(a,b)$! (This also means that the positions of these poles do not depend on $\gamma$!)

Therefore, we have proven the following statement.

***The spectrum of $H_\gamma$ inside $\Omega_\gamma$ represents the scattering-matrix resonances.***

# Conclusion

- For the (analytic) Friedrichs-Faddeev model, we have derived representations that explicitly express the T-matrix and scattering matrix un unphysical energy sheets in terms of these same operators considered exclusively on the physical sheet.

- A resonance on a sheet $\Pi_l$ corresponds to a point $z$ on the physical sheet where the corresponding scattering matrix $S_l(z)$ has eigenvalue zero, that is

$$S_l(z)\mathscr{A} = 0$$

  for some non-zero $\mathscr{A} \in \mathfrak{h}$.

- We have shown that, for the Friedrichs-Faddeev model, the scaling/rotation resonances are exactly the scattering matrix resonances.