# Bound States of Relativistic Nature

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**Abstract**

Bethe–Salpeter equation for massless exchange and large fine structure constant $\alpha > \pi/4$, in addition to Balmer series, provides another (abnormal) series of energy levels which are not given by the Schrödinger equation. So strong field can be created by a point-like charge $Z > 107$. The nuclei with this charge, though available, are far from being point-like that weakens the field. Therefore, the abnormal states of this origin do hardly exist.

We analyze a more realistic case of exchange by a massive particle when the large value of coupling constant is typical for the strong interaction. It turns out that this interaction still generates a series of abnormal relativistic states. The properties of these solutions are studied. Their existence in nature seems to be possible.

**Keywords:** *Bethe–Salpeter equation; massive ladder exchange; relativistic bound states*

## 1 Introduction

The Bethe–Salpeter (BS) equation [1] is a relativistic counterpart of the Schrödinger equation. In the spinless case, for a two-body system, it reads

$$\Phi(k,p) = \frac{i^2}{\left[(\frac{p}{2}+k)^2 - m^2 + i\epsilon\right]\left[(\frac{p}{2}-k)^2 - m^2 + i\epsilon\right]} \int \frac{d^4k'}{(2\pi)^4}\, iK(k,k',p)\, \Phi(k',p), \tag{1}$$

$p$ is the total four-momentum, $k$ is the relative one. The bound state mass sqared is $M^2 = p^2 = (2m - B)^2$ and $B$ is the (positive) binding energy. The kernel in Eq. (1) in the case of exchange by a particle with mass $\mu$ has the form

$$iK(k,k',p) = \frac{i(-ig)^2}{(k-k')^2 - \mu^2 + i\epsilon}. \tag{2}$$

Soon after its derivation, the BS equation was studied by Wick [2] and Cutkosky [3] in the model of two spinless particles interacting by a massless scalar exchange ($\mu = 0$), since known as Wick–Cutkosky model. Solving Eq. (1) in the limit of small binding

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*http://www.ntse.khb.ru/files/uploads/2018/proceedings/Karmanov.pdf.*

energies ($B/m << 1$), these authors reproduced the Coulomb spectrum, i. e., the Balmer series,

$$B_n = \frac{\alpha^2 m}{4n^2}, \tag{3}$$

given also by the Schrödinger equation with the potential $V(r) = -\frac{\alpha}{r}$, where $\alpha = g^2/(16\pi m^2)$. (Wick and Cutkosky [2, 3] used another definition of the coupling constant: $\lambda = \frac{\alpha}{\pi}$). According to Ref. [3], when $\alpha \to 2\pi$, the ground state of mass $M$ determined by Eq. (1) tends to 0. When $\alpha > 2\pi$, there is no physical solution for the ground state: $M^2$ becomes negative.

To solve the BS equation, Cutkosky has represented the BS amplitude for the $S$-wave states in the following integral form:

$$\Phi_n(k,p) = \sum_{r=0}^{n-1} \int_{-1}^{1} \frac{g_n^r(z)\, dz}{[m^2 - \frac{1}{4}M^2 - k^2 - p\cdot k\, z - \imath\epsilon]^{2+n}}, \qquad n = 1, 2, \ldots \tag{4}$$

After substituting Eq. (4) into Eq. (1) and some manipulations, one finds that the functions $g_n^r(z)$ satisfy coupled integral equations with an exception of $g_n^0$ which satisfies a decoupled homogeneous equation. Other functions $g_n^r$, $0 < r \le (n-1)$, are then determined from $g_n^0$ through the remaining equations. Denoting henceforth $g_n^0$ by $g_n$, one obtains the equation for $g_n$,

$$g_n''(z) + \frac{2(n-1)z}{(1-z^2)}\, g_n'(z) - \frac{n(n-1)}{(1-z^2)}\, g_n(z) + \frac{\alpha}{\pi} \frac{1}{(1-z^2)(1-\eta^2+\eta^2 z^2)}\, g_n(z) = 0, \tag{5}$$

where $\eta = \frac{M}{2m} = 1 - \frac{B}{2m}$ and the boundary conditions are $g_n(\pm 1) = 0$. For a given $n$, this homogeneous equation has another infinite spectrum $M_{nk}$ distinct from the ordinary relativistic generalization of the Balmer series, corresponding to bound states $g_{nk}$ with binding energies $B_{nk} = 2m - M_{nk}$ depending on the second integer quantum number $k = 1, 2, 3, \ldots$. In the limit of small binding energies, $B_{nk}$ is independent of $n$, namely:

$$B_{nk} \approx B_k = m \exp\left(-\frac{2\pi^{3/2} k}{\sqrt{\alpha - \pi/4}}\right), \tag{6}$$

For $k = 0$ and arbitrary $n$, the levels are still given by the Balmer series (3) corresponding to the so-called normal ones. The abnormal solutions $g_{nk}(z)$ have $k$ nodes in $z$. The solutions $g_{nk}(z)$ are symmetric in $z \to -z$ for even $k$ and antisymmetric for odd $k$. The corresponding BS amplitudes in the rest frame are symmetric or antisymmetric relative to $k_0 \to -k_0$. It was shown in Ref. [4] that the antisymmetric solutions do not contribute to the $S$-matrix and therefore they are hardly observable. Therefore we will consider the symmetric (normal and abnormal) states only.

To summarize, for the massless exchange, in addition to the Balmer series, the BS equation predicts for each $n$ another series of states with binding energies $B_{nk}$ given by Eq. (6) in the limit $B/m << 1$. These states exist only if $\alpha > \frac{\pi}{4}$. Their binding energies tend to zero when $\alpha \to \frac{\pi}{4}$. They are absent in the spectrum of non-relativistic Schrödinger equation, and therefore they were called "abnormal".

Wick and Cutkosky found analytical solutions of Eq. (5) in the limit $\eta = \frac{M}{2m} \to 1$. We solved this equation numerically for arbitrary $\eta$. The examples of normal and abnormal symmetric solutions $g(z)$ are shown in Figs. 1 and 2 respectively. These solutions correspond to $n = 1$ and differ by the $k$ values: $k = 0$ and $k = 2$.

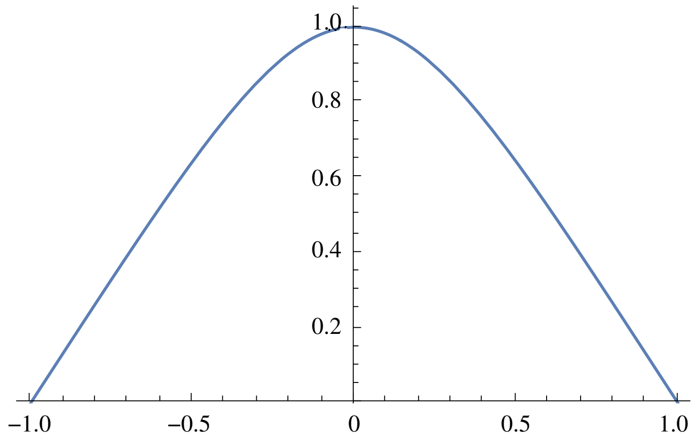


Figure 1: Normal solution $g(z)$ of Eq. (5) ($\mu = 0$) corresponding to $n = 1$, $k = 0$, $B = 0.2$, $\alpha = 1.786$.

The aim of our research is to answer the question: can the abnormal states exist in the nature or not? In the case of the massless exchange considered by Wick and Cutkosky and sketched above, the answer seems to be negative. The required coupling constant $\alpha > \frac{\pi}{4}$ is too large to be reached in practice. Indeed, since the value $\alpha = \frac{1}{137}$ corresponds to $Z = 1$, $\alpha > \frac{\pi}{4}$ corresponds to the charge $Z > \frac{\pi}{4}/(\frac{1}{137}) \approx 107$. Nuclei with this and larger charges, though do not exist in nature, were created in a laboratory ($Z = 107$ corresponds to bohrium). However, they are far from being point-like. Since the charge is distributed in a large volume, the strength of the electric field is reduced. Therefore, to create an abnormal state, one needs even larger (maybe, much larger) value of $Z$. This makes the problem unrealistic.

However, the value $\alpha = \frac{\pi}{4} \approx 0.78$ is normal when dealing with strong interactions. The latters are modeled by a massive particle exchange. Therefore, in our

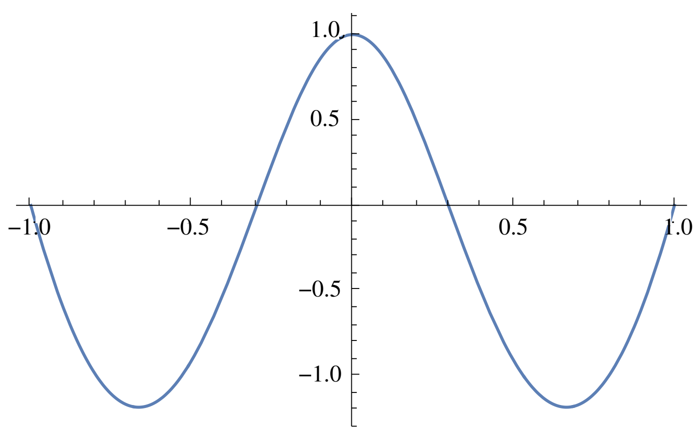


Figure 2: Abnormal symmetric solution $g(z)$ of Eq. (5) ($\mu = 0$) corresponding to $n = 1$, $k = 2$, $B = 0.2$, $\alpha = 17.19$.

research, we will replace the massless exchanged particle by a massive one, and we will study whether or not the abnormal states will still survive. It turns out that the most peculiar properties of the abnormal states in the Wick–Cutkosky model (the existence of the critical coupling constant $\alpha_c = \frac{\pi}{4}$ determining the existence of the abnormal states; the simultaneous appearance and disappearance of infinite series of levels when the coupling constant crosses the critical value) are a consequence of the zero exchanged mass. One could then expect that these properties do not exist anymore in the massive case. However, this does not forbid the existence of the abnormal states at all, though a definite answer requires some research.

## 2 Non-zero exchanged mass

For solving this problem, it is still convenient to use an integral representation for the BS amplitude similar to Eq. (4), namely:

$$\Phi(k,p) = \int_0^\infty d\gamma \int_{-1}^{1} \frac{g(\gamma,z)\,dz}{\left[\gamma + m^2 - \frac{1}{4}M^2 - k^2 - p\cdot k\,z - \imath\epsilon\right]^3}. \tag{7}$$

This representation has been proposed by Nakanishi [5]. To simplify the notations, we omit here the indices $n, k$. In contrast to Eq. (4) for the massless case, the weight function $g(\gamma,z)$ in Eq. (7) depends on an additional variable $\gamma$ and, correspondingly, the integral in Eq. (7) is double. The massless exchange corresponds to a particular situation where the function $g(\gamma,z)$ can be expressed, concerning its $\gamma$ dependence, as a superposition of the delta function and $(n-1)$ its derivatives in $\gamma$:

$$g(\gamma,z) = g_n(\gamma,z) = \sum_{r=0}^{n-1} \delta^{(r)}(\gamma)\, g_n^r(z), \quad n = 1,2,\dots\,. \tag{8}$$

Substituting $\Phi(k,p)$ in the BS equation (1) by its expression (7), one can derive an equation for $g(\gamma,z)$. Some properties of the solutions will be still studied analytically whereas the spectrum and corresponding solutions will be found numerically.

For the ladder BS kernel, the equation for the weight function $g(\gamma,z)$ was firstly derived in Ref. [6], though in a little bit complicated form. For an arbitrary BS kernel, the equation for $g(\gamma,z)$ was derived in Ref. [7], though in the form containing integrals in both sides of equation. It reads:

$$\int_0^\infty \frac{g(\gamma',z)\,d\gamma'}{\left[\gamma' + \gamma + z^2m^2 + (1-z^2)\kappa^2\right]^2} = \int_0^\infty d\gamma' \int_{-1}^{1} dz'\; W(\gamma,z;\gamma',z')\; g(\gamma',z'), \tag{9}$$

where $\kappa^2 - m^2 - \frac{1}{4}M^2$. In the canonical form,

$$g(\gamma,z) = \int_0^\infty d\gamma' \int_{-1}^{1} dz'\; \mathcal{V}(\gamma,z;\gamma',z')\; g(\gamma',z'), \tag{10}$$

for the ladder BS kernel, the equation for $g(\gamma,z)$ was derived in Ref. [8]. In this work, the expression for the kernel $\mathcal{V}(\gamma,z;\gamma',z')$ corresponding to the ladder BS kernel, was found.

It was noticed in Ref. [9] that the l.h.s. of Eq. (9) is the generalized Stieltjes transform which can be inverted analytically. In this way, the equation for $g(\gamma, z)$ in the canonical form valid for arbitrary BS kernel, was derived. For the ladder BS kernels, the kernels $\mathcal{V}(\gamma, z; \gamma', z')$ in Eq. (10) found in Ref. [9] and [8], coincide with each other. A useful research of the non-relativistic limit of the BS equation was done in Ref. [10].

We will analyze the equation in the form (10) with the kernel derived in Ref. [8] from the ladder kernel (2). This kernel reads

$$\mathcal{V}(\gamma, z; \gamma', z') = +\frac{\alpha m^2}{2\pi} \times \begin{cases} h(\gamma, -z; \gamma', -z') & \text{if } -1 \le z' \le z \le 1, \\ h(\gamma, z; \gamma', z') & \text{if } -1 \le z \le z' \le 1, \end{cases} \tag{11}$$

with the function

$$h(\gamma, z; \gamma', z') = \theta(\eta)\, P(\gamma, z, \gamma', z') + Q(\gamma', z'), \tag{12}$$

where

$$P(\gamma, z, \gamma', z') = \frac{B}{\gamma A \Delta} \frac{1+z}{(1+z')} - C(\gamma, z, \gamma', z')$$

with

$$A(\gamma', z') = \frac{1}{4} z'^2 M^2 + \kappa^2 + \gamma', \qquad B(\gamma, z, \gamma', z') = \mu^2 + \gamma' - \gamma \frac{1+z'}{1+z},$$

$$C(\gamma, z, \gamma', z') = \int_{y_-}^{y_+} \chi(y)\, dy, \qquad Q(\gamma', z') = \int_0^\infty \chi(y)\, dy,$$

$$\Delta(\gamma, z, \gamma', z') = \sqrt{B^2 - 4\mu^2 A}.$$

The functions $C$ and $Q$ contain the function

$$\chi(y) = \frac{y^2}{\left[y^2 + A + y(\mu^2 + \gamma') + \mu^2\right]^2}$$

and the integration limits in $C$ are given by $y_\pm = \frac{-B \pm \Delta}{2A}$. The argument $\eta$ of the $\theta$-function in the first term of Eq. (12) is

$$\eta = -B - 2\mu\sqrt{A} = \gamma \frac{1+z'}{1+z} - \mu^2 - \gamma' - 2\mu \sqrt{\frac{1}{4} z'^2 M^2 + \kappa^2 + \gamma'}.$$

The results of solving numerically Eq. (10) with the parameters $\mu = 0.15$ and $B = 0.2$ are displayed in Fig. 3. The coupling constant $\alpha = 2.1$ and corresponds to the "normal" state. They have been obtained in a recent work [11] by using the same spline techniques as in Ref. [7]. The Nakanishi weight function $g$ has been computed by several authors in the past either by solving Eq. (9) or its equivalent normal form of Eq. (10). None of them put in evidence a striking behavior of this quantity manifested in Fig. 3 — it is a step-like function on variable $\gamma$ but has a flat behavior in some domain as a function of variable $z$. The numerical difficulties in finding solution appear, on one hand, because of the $g$ numerical instabilities related to the $\epsilon$-trick introduced in Ref. [7], and, on the other hand, because of describing a flat behavior by a Gaussian-like basis expansion employed in Refs. [8, 12–14]. This behavior has been also proved analytically in Ref. [11] and will be discussed below in Section 4.

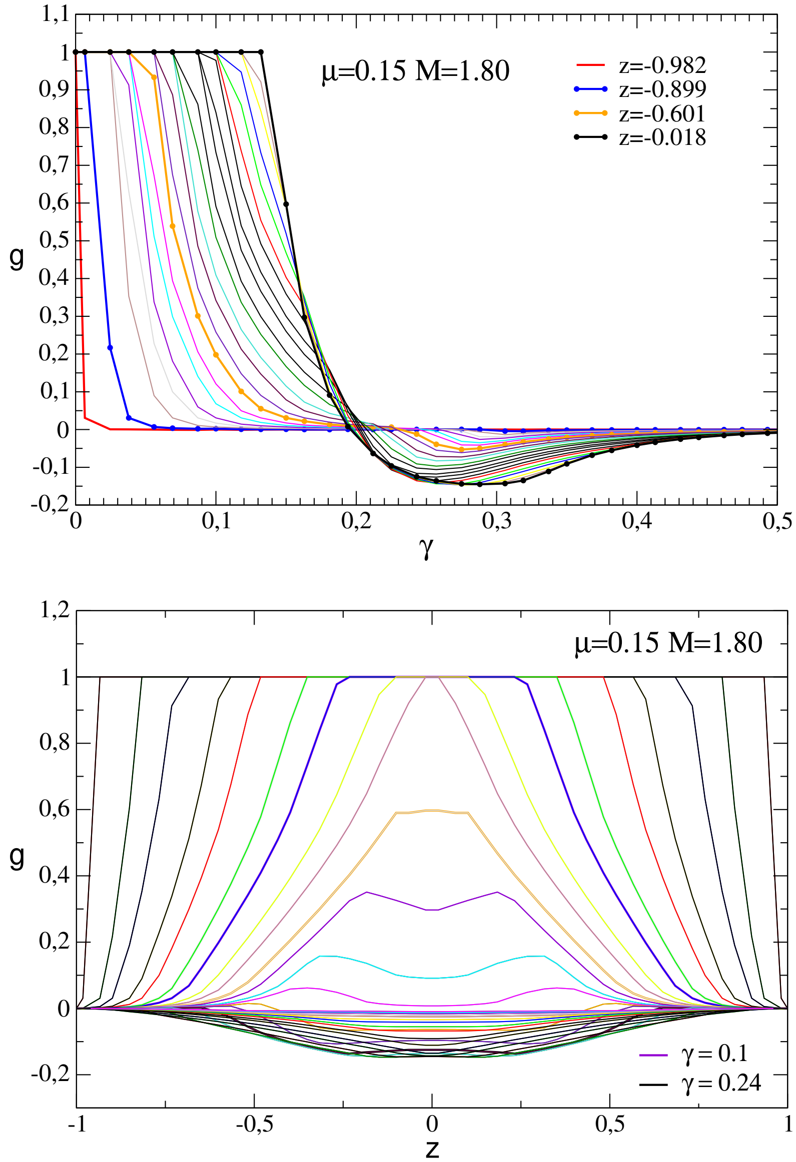


Figure 3: Nakanishi weight function $g(\gamma, z)$ corresponding to $m = 1$, $\mu = 0.15$, $B = 0.2$ and $\alpha = 2.1$ as a function of $\gamma$ for a fixed $z$ (top) and as a function of $z$ for a fixed $\gamma$ (bottom).

## 3 Non-relativistic limit

The "relativistic world" differs from the non-relativistic one by the existence of the limiting value of speed of any object or signal, which is identified with the speed of light $c$. Calculating via a relativistic equation the binding energy corresponding to a normal state and taking the limit $c \to \infty$, we should obtain the non-relativistic binding energy. The abnormal states, not existing in the non-relativistic limit, should disappear when $c \to \infty$. The relativistic equations presented above implied $c = 1$. To study the limit $c \to \infty$, we should now restore the speed of light $c$ in these equations.

The strategy is the following. We should introduce $c$ in the parameters which are used as an input in the equation. This, of course, automatically has an influence on the parameters which are the "output" (found from the equation), therefore we should leave their value untouched. This means that we should replace $m$ by $mc^2$. As for the total mass, it is not an independent parameter (not an input), it is expressed as $M = 2mc^2 - B$. Therefore one *should not* make the replacement $M \to Mc^2$. The same is valid for $B$ since it is also not an input, but it is found from the equation (already containing $mc^2$) as an eigenvalue. Therefore one also *should not* make the replacement $B \to Bc^2$, one should keep instead the binding energy $B$ as it is. Since the coupling constant in QED is $\alpha = e^2/(\hbar c)$, the $c$ value should appear in the coupling explicitly. Therefore $\alpha$ should be replaced by $\alpha/c$.

In the last replacement, there was no any reference to the mass of the exchange particle. Therefore it is valid not only in QED, but also in the Yukawa model with a massive particles exchange. A subtle point is the replacement of the exchanged mass $\mu$. The ladder exchange results in the Yukawa potential with the factor $\sim \exp(-\mu r)$. Restoring $c$ in this factor, we get $\exp\left(-\frac{\mu c^2 r}{\hbar c}\right)$. We get a zero-range potential in the limit $c \to \infty$, the Yukawa potential shrinks to a delta-function. However, in the present research, we study how the energies found from a relativistic equation are transformed into the energies determined by the Schródiger equation with a given potential $V(r)$, and we are not interested in the effects resulting from the variation of $V(r)$ with $c$. The shrink is avoided if we replace $\mu \to \mu c$, not $\mu \to \mu c^2$.

With these replacements made in the kernel $\mathcal{V}$, we solve Eq. (10) numerically and, varying $c$, we study the behavior of two energy levels. More precisely, for a fixed binding energy $B$, we study the behavior of the coupling constant $\alpha$ as a function of $c$ in the interval $1 \leq c \leq 10$. We use $m = 1$, $\mu = 0.15$ and $B = 0.1$. For one of the states, which we associate with the "normal" solution, we found $\alpha(c = 1) \approx 1.45$. For another state, which we associate with the "abnormal" solution, we found $\alpha(c = 1) \approx 10$. The results for $\alpha(c)$ for these two states are shown in Figs. 4 and 5. These curves have opposite behaviors (decreasing and increasing) as functions of $c$.

We see in Fig. 4 that in the non-relativistic limit ($c \to \infty$) $\alpha$ decreases and tends to the limiting finite value $\alpha \approx 0.9$. This value is just the coupling constant of the Yukawa potential providing the binding energy $B = 0.1$ in the Schrödinger equation. Therefore we associate this solution with the "normal" one which has the non-relativistic limit. The decrease of $\alpha$ with $c$ seen at Fig. 4 can be easy explained qualitatively. As it was noticed in many papers, the relativistic effects added to the non-relativistic dynamics, result in an effective repulsion. Therefore, when we go to the non-relativistic limit ($c$ increases), we decrease this repulsion. Hence, we need a smaller coupling constant $\alpha$ to keep the fixed value of the binding energy $B = 0.1$.

According to the curve $\alpha(c)$ shown in Fig. 5, the value of $\alpha$ increases with $c$, at least in the interval $1 \leq c \leq 10$. The disappearance of the abnormal states as $c$ increases means that the corresponding energy levels are "pushed out" into the continuum spectrum. That is, they move up and cross the value $B = 0$. To prevent this movement and to keep these levels at a constant value, say, at $B = 0.1$, like in Fig. 5 at $c = 1$, one should increase the attraction. Hence, when $c$ increases, we need a larger coupling constant $\alpha$ as is observed in Fig. 5. Therefore we associate this

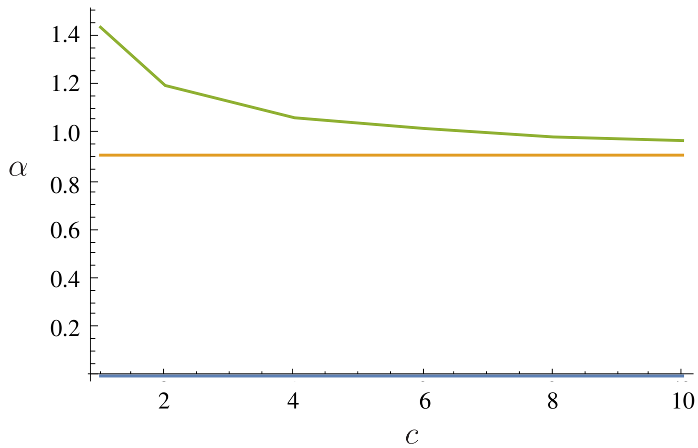


Figure 4: The coupling constant $\alpha$ vs the speed of light $c$ for the parameters $m = 1$, $\mu = 0.15$, $B = 0.1$. The decreasing line is found from the relativistic Eq. (10) for the "normal" solution; the horizontal line is the limiting value of this decreasing line, the coupling constant for the Yukawa potential in the Schrödinger equation.

solution with the "abnormal" one.

These results demonstrate the existence of the abnormal states in the solution of the BS equation with the massive ladder kernel (we assume that the qualitative behavior of $\alpha$ as a function of $c$ can be extrapolated to larger $c$ values). At least one of them is found and the corresponding $\alpha(c)$ function is shown in Fig. 5.

Coming back to the zero-mass exchange, we can make the replacements $m \to mc^2$ and $\alpha \to \alpha/c$ in the Eq. (6) describing the binding energy $B_k$. Then, solving this

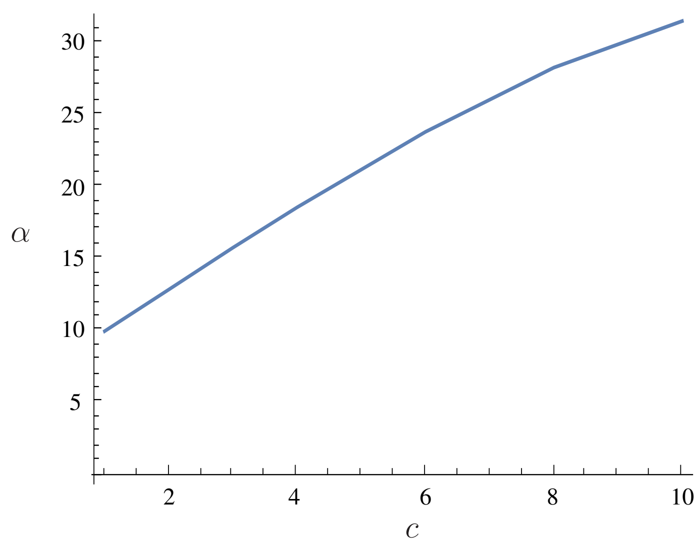


Figure 5: The coupling constant $\alpha$ vs the speed of light $c$ for the parameters $m = 1$, $\mu = 0.15$, $B = 0.1$ for the "abnormal" solution.

equation with respect to $\alpha$, we find:

$$\alpha = c\left(\frac{\pi}{4} + \frac{4\pi^3 k^2}{\log^2 \frac{B_k}{mc^2}}\right). \tag{13}$$

This formula gives an analytical example of the dependence $\alpha(c)$ which has no finite limit at $c \to \infty$, that serves as an undoubted property of the abnormal state.

We remind that for the massless exchange, there exists another criterion to select an abnormal state: it is the existence of nodes in the solution $g(z)$ (see Fig. 2). Though for distinguishing an abnormal state it is sufficient to fit only one of these two criterions, it is useful to establish both. For the massive exchange, we discussed so far only one criterion for the selection of an abnormal solution: the absence of a finite limit of $\alpha$ as $c \to \infty$. Below we will formulate another criterion, also based on an analysis of nodes of the solution $g(\gamma, z)$.

## 4 Properties of the $z$-dependence of the solution $g(\gamma, z)$

In the case of massless exchange discussed in Section 1, a normal solution $g(z)$ has no nodes (see Fig. 1). However, for massive exchange, this property cannot be used to distinguish a normal solution. In Fig. 6 we show the solution $g(\gamma, x)$ (with the parameters $m = 1$, $\mu = 0.15$, $B = 0.1$, $\alpha = 1.4375$) for the fixed value $\gamma = 0.17$ as a function of $x = \frac{1}{2}(1+z)$. Instead of $z$, we introduced for convenience the new variable $x$ varying in the limits $0 \le x \le 1$. This is a normal solution since the dependence $\alpha(c)$ corresponding to this solution, is shown in Fig. 4.

In spite of the fact that this solution is normal, it has nodes as a function of $x$ for a fixed $\gamma$. However, it turns out that the behavior of $g(\gamma, x)$ is qualitatively different in different parts of the domain of its definition. There is an area where $g(\gamma, x) = const$.

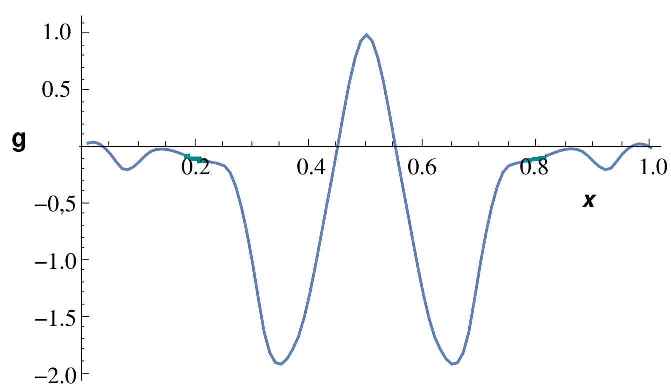


Figure 6: Normal solution $g(\gamma = 0.17, x)$ of Eq. (10) with the kernel of Eq. (11) vs $x = \frac{1}{2}(1+z)$ for the parameters $m = 1$, $\mu = 0.15$, $B = 0.1$, $\alpha = 1.4375$ at the fixed value of $\gamma = 0.17$.

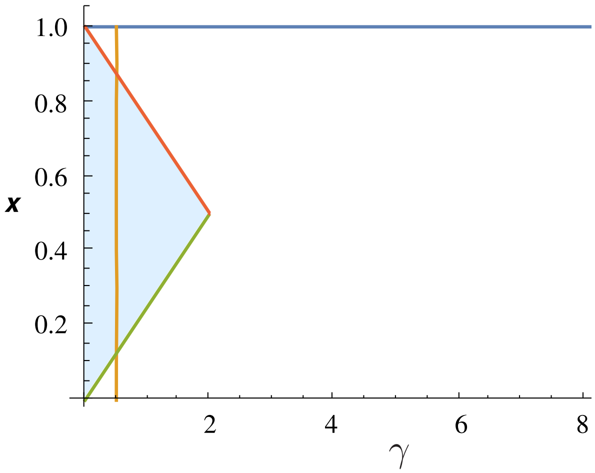


Figure 7: Domain of definition of the function $g(\gamma, x)$. Shaded area is the domain defined by Eq. (14) where $g(\gamma, x) = const.$

This area is defined (up to a factor) by:

$$\gamma \leq \gamma_0(x) \sim \left\{ \begin{array}{ll} \mu x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \mu(1-x) & \text{if } \frac{1}{2} \leq x \leq 1. \end{array} \right. \tag{14}$$

The domain ($0 \leq \gamma < \infty$, $0 \leq x \leq 1$) is shown in Fig. 7, the shaded area corresponds to Eq. (14).

Let us fix the $\gamma = 0.005$ from the area defined by Eq. (14) and consider the $x$ dependence of $g(\gamma = fixed, x)$. This corresponds to the variation of $x$ along the vertical line crossing the triangle in Fig. 7. Solving numerically Eq. (10) for the normal solution of the type shown in Fig. 6 for $\gamma = 0.17$, we now obtain Fig. 8. We see that in the interval $x_1 < x < x_2$ where the vertical line in Fig. 7 is inside the shaded triangle, the function $g(\gamma = 0.005, x)$ is indeed constant, as expected. Outside

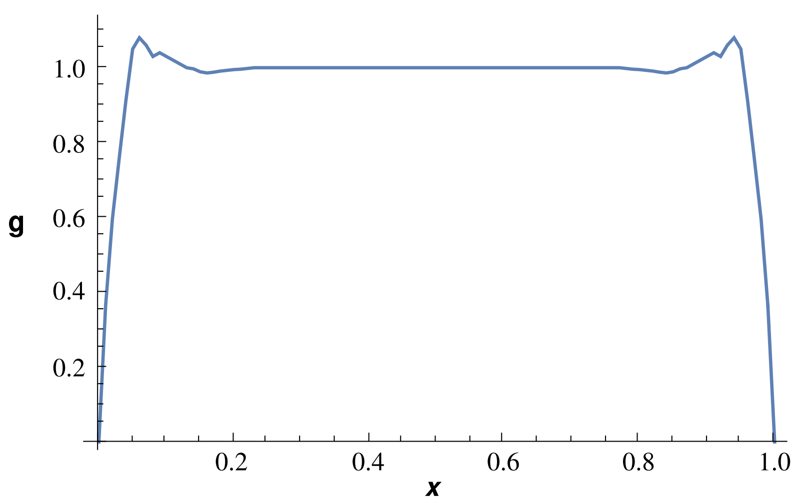


Figure 8: Normal solution $g(\gamma = 0.005, x)$ of Eq. (10) with the kernel of Eq. (11) vs $x$ for the same parameters as in Fig. 6.

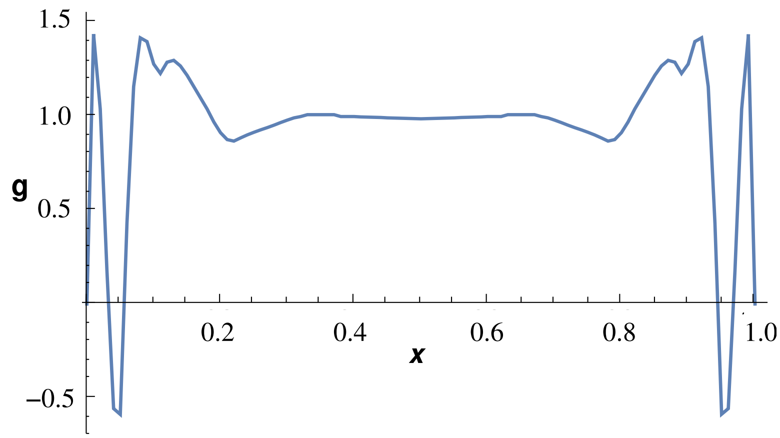


Figure 9: Abnormal symmetric solution $g(\gamma = 0.005, x)$ of Eq. (10) with the kernel of Eq. (11) vs $x$ for the same parameters as in Figs. 6 and 8 except $\alpha = 11.2859$.

this interval, when $x$ is close to 0 or to 1, $g(\gamma = 0.005, x)$ varies. However, it has no nodes. In this respect, the behavior of the normal solution $g(\gamma = 0.005, x)$ is analogous to the behavior of normal $g(z)$ for the massless exchange. The latter has nodes nowhere except for the points $x = 0$ and 1, where the nodes are imposed by the boundary conditions.

The symmetric abnormal solution for the same parameters (though, of course, for a different binding energy) is shown in Fig. 9. It is still constant when $x$ is inside the domain defined by Eq. (14) and has nodes outside this domain, in contrast to the normal solutions. Like for the massless exchange, this gives us another criterion $\big($in addition to the limit $\alpha(c \to \infty)\big)$ to distinguish, in the case of the massive exchange, the abnormal solutions from the normal ones.

We emphasize that these results are based on the numerical calculations. It would be useful to derive them analytically.

# 5 Conclusion

Like the Dirac equation predicting antiparticles, the BS equation predicts bound states having a purely relativistic origin: these are the so-called “abnormal” states, not given by the Schrödinger equation.

We have found that such states, previously obtained in the Wick–Cutkosky model (scalar massless exchange) in the case of a large coupling constant, exist also for the interaction provided by massive exchange with values of the coupling constant typical for the strong interaction. It is worth conjecturing that these states could be manifested in some processes in nature. One should analyze from this point of view systems which is difficult to describe as ordinary bound states and and which require exotic speculations. Maybe, some of these systems are “abnormal” ones.

For a deeper understanding of the abnormal states, it would be useful to calculate corresponding electromagnetic form factors, to compare them with the “normal” form factors and to calculate also the transition form factors of the type normal $\leftrightarrow$ abnormal states.

A clarification of the content of the "abnormal" state vector, i. e., of the contributions to its norm of the Fock components with different numbers of particles, is still an intriguing problem. Preliminary results look as follows. For the normal state, when the binding energy tends to zero, the contribution of the two-body sector dominates in the norm of the state vector. On the contrary, for the abnormal state, when the binding energy tends to zero ($\alpha \to \frac{\pi}{4}$), the contribution of the two-body sector to the norm of the state vector decreases.

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