Nonperturbative Calculations in the Light-Front Field Theory

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Abstract

A nonperturbative approach to field theory based on the decomposition of the state vector in Fock components, and on the covariant formulation of light-front dynamics, together with the Fock sector dependent renormalization scheme, is briefly reviewed. The approach is applied to the calculation, in the framework of three-body Fock space truncation, of the fermion electromagnetic form factors in the Yukawa model (in particular, of anomalous magnetic moment). Once the renormalization conditions are properly taken into account, the anomalous magnetic moment does not depend on the regularization scale when the latter is much larger than the physical masses.

Keywords: *Light-front dynamics; Yukawa model; non-perturbative renormalization; electromagnetic form factors*

1 Introduction

In the quantum field theory, due to the particle creation and annihilation, the number of particles in a system is not fixed and the state vector is a superposition of the states (Fock sectors) with different numbers of particles:

$$|p\rangle = \sum_{n=1}^{\infty} \int \psi_n(k_1, \ldots, k_n, p)\, |n\rangle\, D_k. \tag{1}$$

ψ_n is the n-body wave function and D_k is an integration measure. In the cases when we can expect that the decomposition (1) converges rapidly enough, we can make truncation, that is replace the infinite sum in (1) by a finite one. Then, substituting the truncated state vector in the eigenvector equation

$$H\,|p\rangle = M\,|p\rangle,$$

we obtain a system of integral equations of finite dimension for the Fock components ψ_n which can be solved numerically. With the decomposition (1), the normalization condition for the state vector $\langle p'|p\rangle = 2\,p_0\,\delta^{(3)}(\mathbf{p}' - \mathbf{p})$ writes as

$$\sum_{n}^{\infty} I_n = 1, \tag{2}$$

where I_n is the contribution of the n-body Fock sector to the norm.

In this way we do not require the smallness of the coupling constant. The approximate (truncated) solution is non-perturbative. This is the basis of a non-perturbative approach which we developed, together with J.-F. Mathiot and A. V. Smirnov, in a series of papers [1–5] (see for review [6]).

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The main difficulty on this way is to ensure cancellation of infinities. In a perturbative approach, for a renormalizable field theory, this cancellation is obtained as a by-product after renormalization in any fixed order of coupling constant. For the cancelation it is important to take into account the complete set of graphs in a given order. Omitting some of these graphs destroys the cancellation and the infinities survive after renormalization. Namely that happens after the truncation: though the truncated solution can be decomposed in infinite series in terms of the coupling constant, it does not contain the complete set of perturbative graphs in any given order. Therefore the standard renormalization scheme does not eliminate the infinities. To provide cancellation of infinities, the sector-dependent renormalization scheme has been proposed [7]. This scheme, in which the values of the counter terms are precised from sector to sector according to unambiguously formulated rules, was developed in detail in Ref. [3] and applied to calculation of electromagnetic form factors in Refs. [4, 5]. Below we will give its brief review and present some results obtained in this approach.

We discuss the convergence of the decomposition (1) in Section 2. The sector dependent renormalization scheme is briefly described in Section 3. It is applied to calculation of anomalous magnetic moment of fermion in Yukawa model. The antifermion degrees of freedom are taken into account in Section 4. The results are summarized in Section 5.

2 On convergence of the Fock decomposition

We work in the light-front dynamics (LFD) [8–10], more precisely, in its explicitly covariant version [8, 9]. In four-dimensional space, the state vector (1) is defined on the light-front plane of a general orientation $\omega\cdot x = 0$, where ω is an arbitrary four-vector restricted by the condition $\omega^2 = 0$ [8, 9]. The traditional form of LFD [10] is recovered by using $\omega = (1, 0, 0, -1)$.

As mentioned, the truncation of the Fock decomposition can be efficient if the decomposition (1) converges rapidly enough. The convergence depends, of course, on the nature of a system under consideration. If this system is dominated by a finite number of degrees of freedom (like hadrons in quark models), then the decomposition (1) is determined with a good accuracy by a finite number of the components. Notice that these "degrees of freedom", e. g., quarks as basis of decomposition (1), may be some effective dressed constituents.

The convergence of the Fock decomposition was estimated [11] in the explicitly solvable Wick–Cutkosky model [12]. This model corresponds to spineless massive particles with equal masses m interacting by spineless massless exchange. One can find the two-body Bethe–Salpeter amplitude. The requirement for the electromagnetic form factor $F(Q^2 = 0) = 1$ fixes the normalization of the Bethe–Salpeter amplitude. On the other hand, projecting the Bethe–Salpeter amplitude on the light-front plane, we find the two-body Fock component of the state vector (1). Its normalization integral is not unity but gives the two-body contribution to the full normalization. One can also estimate the valence three-body contribution. We chose the parameters maximally unfavorable for dominance of a few-body sector. Namely, the coupling constant is very strong, $\alpha = 2\pi$, that corresponds to the limiting case when the binding energy in the Wick–Cutkosky model, $E_b = -2m$, compensates the full mass of the system. The strong coupling constant increases contributions of higher orders, i. e., of many-body components. In addition, since the exchange particles are massless, they can be easy created. The result for different contributions [11] is given in Table 1.

We see that even in this unfavorable case, the Fock states with 2 and 3 particles contribute 90% to the normalization integral. This would give the 10% accuracy in calculation of observables, say, of the electromagnetic form factor.

Table 1: Contributions of the Fock sectors with the number of particles $n=2$, $n=3$ and $n\geq 4$ ($I_{n\geq 4}=\sum_{n=4}^{\infty} I_n$) to the full normalization integral $I=\sum_{n=2}^{\infty} I_n=1$ of the state vector for $M=0$ ($\alpha=2\pi$).

I_2	I_3	$I_{n\geq 4}$	$I_2+I_3+I_{n\geq 4}$
0.643	0.257	0.100	1

3 Fock sector dependent renormalization scheme

We will find the state vector of the Yukawa model Hamiltonian containing the fermion field ψ and the scalar field ϕ with the interaction vertex $g_0\bar{\psi}\psi\phi$. For regularization, we include in the Hamiltonian the Pauli–Villars fields (one fermion and one boson), which, after that, appear in the basis of decomposition (1). Since our formalism is explicitly covariant, the spin structure of the wave function ψ_n is easy found. It should incorporate the ω-dependent components. Therefore the spin structure of the two-body component in the Yukawa model reads:

$$\bar{u}(k_1)\,\Gamma_2 u(p)=\bar{u}(k_1)\left[b_1+\frac{M\,\not{\omega}}{\omega\cdot p}\,b_2\right]u(p). \tag{3}$$

where $\not{\omega}=\omega_\mu\gamma^\mu$. The coefficients b_1 and b_2 are scalar functions determined by dynamics. In LFD they depend on well-known variables $k_\perp$, x: $b_{1,2}=b_{1,2}(k_\perp,x)$.

System of equations for the Fock components in the truncation $N=3$ is graphically shown in Fig. 1. One can exclude the three-body component and obtain a reduced system of equations which includes the one- and two-body components only. It is shown in Fig. 2. In comparison to the equations of Fig. 1, we included in the equations of Fig. 2 another counter term Z_ω discussed below in this section. Namely this reduced system of eight equations for two two-body spin components, for physical and Pauli–Villars fermions and bosons ($2\times 2\times 2=8$), was solved numerically. It contains also a one-body component Γ_1, but it is a constant which can be found from the normalization condition. The limit of the fermion Pauli–Villars mass $m_1\to\infty$ was taken analytically, whereas the limit of the boson Pauli–Villars mass $\mu_1\to\infty$ was taken numerically.

The renormalization condition, as always, means that one should express the bare coupling constant g_0 and the fermion mass counter term δm via the physical coupling

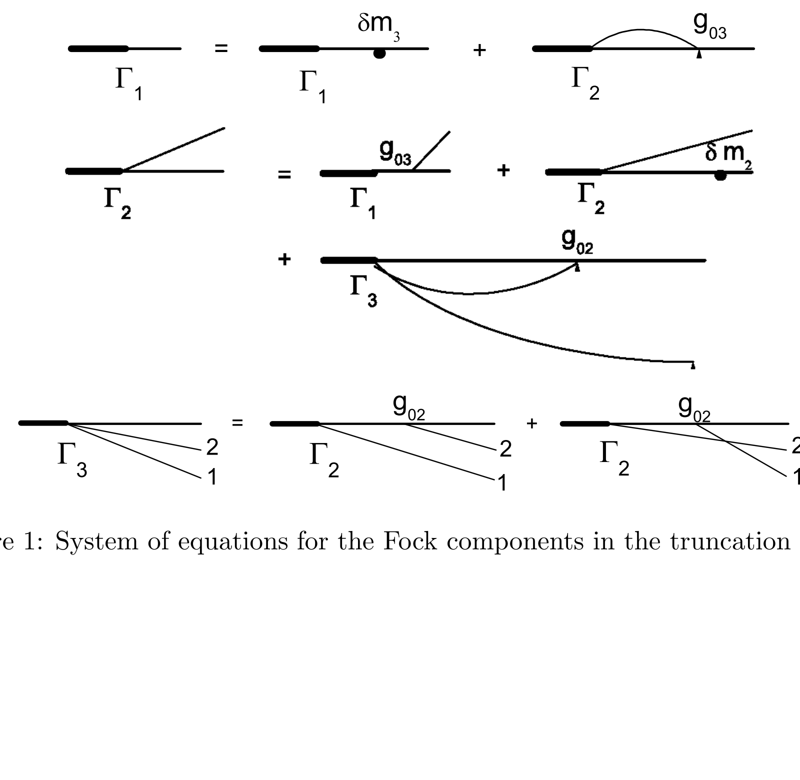

Figure 1: System of equations for the Fock components in the truncation $N=3$.

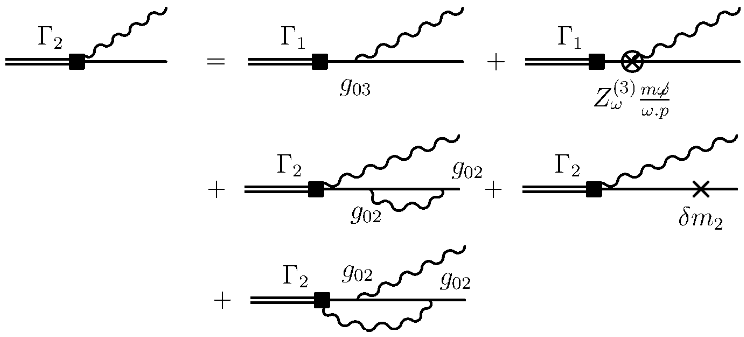

Figure 2: Equation for the two-body component.

constant and mass. In perturbation theory, as mentioned, this leads, as a by-product, to cancellation of infinities. In the non-perturbative approach the strategy is, in principle, the same, however, because of truncation, the infinities are not cancelled. Therefore we use the sector-dependent renormalization scheme [3]. The fermion mass counter term δm, and the bare coupling constant g_0, are thus extended to a sequence:

$$g_0 \rightarrow g_{0l}, \tag{4}$$

$$\delta m \rightarrow \delta m_l, \tag{5}$$

The index l runs through the Fock sectors with $l = 1, 2, ..., N$. The quantities g_{0l} and δm_l are calculated by solving the systems of equations for the vertex functions in the $N = 1$, $N = 2$, $N = 3$, ... approximations successively. That is, we start with the first non-trivial case $N = 2$ and find g_{02}, δm_2. In the case $N = 3$, the parameters g_0, δm_0 may appear in two ways. Namely: (i) Via the two-body sector which enters the three-body equations, as it happens, for example, in the last line in Fig. 1. In this case we use for them already found values of g_{02} and δm_2. (ii) Via the one-body sector which also enters the three-body equations, as it happens, for example, in the first line in Fig. 1. Then these parameters are the new ones: g_{03}, δm_3, which did not appear in the previous $N = 2$ truncation. They are found from the renormalization conditions in the three-body sector. This procedure continues for increasing N. As an example, system of equations for the next $N = 4$ truncation is shown in Fig. 3.

The renormalization condition for the coupling constant g_0 is a relation between the two-body components Γ_2 (containing g_0) and the physical coupling constant g. In order to fix this relationship, one needs to take into account the normalization factors of the external legs of the two-body vertex function. These normalization factors do

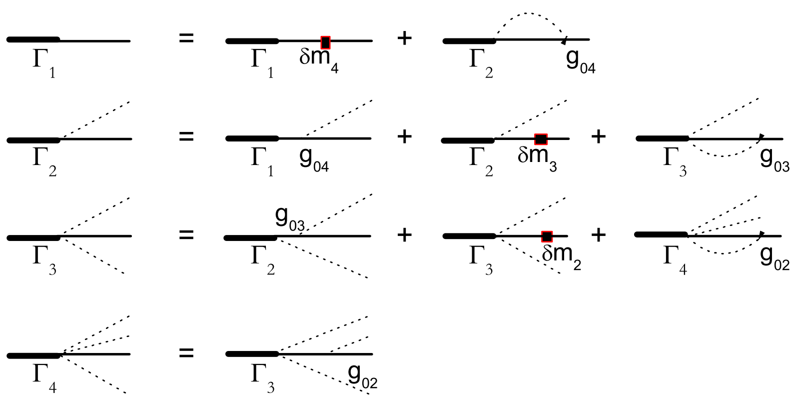

Figure 3: System of equations for the Fock components in the truncation $N = 4$.

also depend on the order of truncation of the Fock space. In the Yukawa model, this relationship reads [6]:

$$\Gamma_2^{(N)}(s_2 = M^2) = g\sqrt{I_1^{(N-1)}}. \tag{6}$$

Here $\Gamma_2^{(N)}$ is the two-body Fock component found in the truncation N whereas $I_1^{(N-1)}$ is the one-body normalization integral (for the fermion state) calculated in the previous $N-1$ truncation. We omit the corresponding boson factor since we do not consider here fermion loops and the boson self-energy (quenched approximation).

The condition (6) has an important consequence: the two-body vertex function, depending according to Eq. (3) on ω, at the value $s_2 = M^2$ should be independent of the orientation ω of the light-front plane. With the spin decomposition (3), this implies that the component b_2 at $s_2 = M^2$ should be identically zero:

$$b_2^{(N)}(s_2 = M^2) \equiv 0. \tag{7}$$

If Eq. (7) is satisfied, Eq. (6), in the quenched approximation, turns into

$$b_1^{(N)}(s_2 = M^2) \equiv g\sqrt{I_1^{(N-1)}}. \tag{8}$$

While the property (7) is automatically fulfilled in the case of the two-body Fock space truncation provided one uses a rotationally invariant regularization scheme [6], this is not guaranteed for higher order truncations. The ω-dependence of the off-shell vertex Γ_2, Eq. (3), is allowed even for the exact solution, but it must completely disappear on the energy shell. Because of approximations, the latter does not happen automatically.

Another consequence of the truncation of the Fock space is the fact that the components $b_{1,2}(s_2 = M^2)$ are not constants. Indeed, $b_{1,2}$ depend on two kinematical variables $k_\perp, x$. The on-shell condition

$$s_2 \equiv \frac{k_\perp^2 + m^2}{1-x} + \frac{k_\perp^2 + \mu^2}{x} = M^2 \tag{9}$$

can be used to fix one of the two variables, say $k_\perp$, in the non-physical domain (for $M = m$):

$$k_\perp = k_\perp^*(x) \equiv i\sqrt{x^2 m^2 + (1-x)\mu^2}, \tag{10}$$

so that $b_{1,2}(s_2 = M^2) \equiv b_{1,2}(k_\perp^*(x), x)$ depends on x, whereas the conditions (7) and (8) should be valid identically, i. e. for any value of x.

In order to enforce the condition (7), we introduce an appropriate counterterm which depends explicitly on the four-vector ω. It is shown by cross in the first line of Fig. 2. It originates from the following additional term introduced in the interaction Hamiltonian:

$$\delta H_\omega^{int} = -Z_\omega \bar{\psi}' \, \frac{m \not\omega}{i\omega\cdot\partial} \, \psi' \varphi', \tag{11}$$

where Z_ω is just the new counterterm, $\psi'(\varphi')$ is the fermion (scalar boson) field being a sum of the corresponding physical and Pauli–Villars components: $\psi' = \psi + \psi_{PV}$ and similarly for φ'; $1/(i\omega\cdot\partial)$ is the reversal derivative operator. The enforcement of the condition (7) for any x by an appropriate choice of the counterterm Z_ω implies that Z_ω should *a priori* depend on x, i. e. $Z_\omega = Z_\omega(x)$. It induces also an unique dependence of $g_{0N} = g_{0N}(x)$ as a function of the kinematical variable x.

The fact that, in order to satisfy the renormalization conditions, the bare parameters must depend on the kinematical variable x, is crucial to obtain results which are finite after the renormalization procedure in the truncated Fock space. The stability of our results relative to the value of the regularization scale, if the latter reasonably exceeds the physical masses, is confirmed numerically with high precision (see Fig. 4 below).

At first glance, the x-dependence of the bare parameters seems unusual. However, it is a natural consequence of the truncation. Of course, the bare parameters in the fundamental non-truncated Hamiltonian are true constants. After truncation, the initial Hamiltonian is replaced by a finite matrix which acts now in a finite Fock space. But it turns out that the modification of the Hamiltonian is not restricted to a simple truncation. Indeed, to preserve the renormalization conditions, the bare parameters in this *finite* matrix become x-dependent. This x-dependence cannot be derived from the initial Hamiltonian. It appears only after the Fock space truncation.

Our truncated Hamiltonian with the x-dependent bare parameters is a self-consistent approximation to the initial fundamental Hamiltonian. One expects that the approximation becomes better when the number of Fock components increases. At the same time, the x-dependence of the bare parameters should become weaker. An indication of this behavior, when one adds the antifermion contribution, is found in Ref. [5] and is demonstrated below in Section 4. We emphasize that there is no any ambiguity in finding the bare parameters, in spite of their x-dependence. These functions of x are completely fixed by the renormalization conditions.

Using this scheme, in the three-body truncation (up to 1 fermion + 2 bosons), we calculated [4, 5] the fermion electromagnetic form factors $F_1(Q^2)$ and $F_2(Q^2)$. In all computations, we used the physical particle masses $m = 0.938$ and $\mu = 0.138$ reflecting the characteristic nuclear physics mass scales. Each physical quantity was calculated for three values of the physical coupling constant $\alpha = g^2/4\pi = 0.5$, 0.8, and 1.0.

The value $F_2(0)$ is the anomalous magnetic moment (AMM). It is shown in Fig. 4 as a function of the Pauli–Villars boson mass μ_1. Each of the two- and three-body Fock sector contributions to the AMM essentially depends on μ_1, while their sum is stable as μ_1 becomes large enough. Note that using x-dependent bare parameters removes μ_1-dependence of the AMM observed in Ref. [4] already for $\alpha \sim 0.5$, even for larger coupling constants.

As it was explained, we took the limit $m_1 \to \infty$ analytically and then the limit of large μ_1 numerically. For a test of stability of our calculations, we compare in Table 2 the numerical results for AMM obtained in two different orders of limits of large Pauli–Villars masses. The AMM is considered as a function of the Pauli–Villars mass which is kept finite (m_1, if the limit $\mu_1 \to \infty$ has been taken, and vice versa). For convenience of the comparison, we took the same sets of finite mass values for Pauli–Villars boson and fermion.

If each of the finite Pauli–Villars masses is much larger than all physical masses, the values of the AMM, obtained in both limits, coincide within the computational accuracy (about 0.2%), as it should be if the renormalization procedure works properly. We can thus choose any convenient order of the infinite Pauli–Villars mass limits. Since the equations for the Fock components are technically simpler in the

Table 2: The anomalous magnetic moment calculated for $\alpha = 0.8$ in the two different limits of the Pauli–Villars masses.

Pauli–Villars mass kept finite (μ_1 or m_1)	AMM when $m_1 \to \infty$	AMM when $\mu_1 \to \infty$
5	0.1549	0.1454
10	0.1641	0.1630
25	0.1690	0.1704
50	0.1702	0.1715
100	0.1706	0.1716
250	0.1708	0.1714
500	0.1709	0.1713

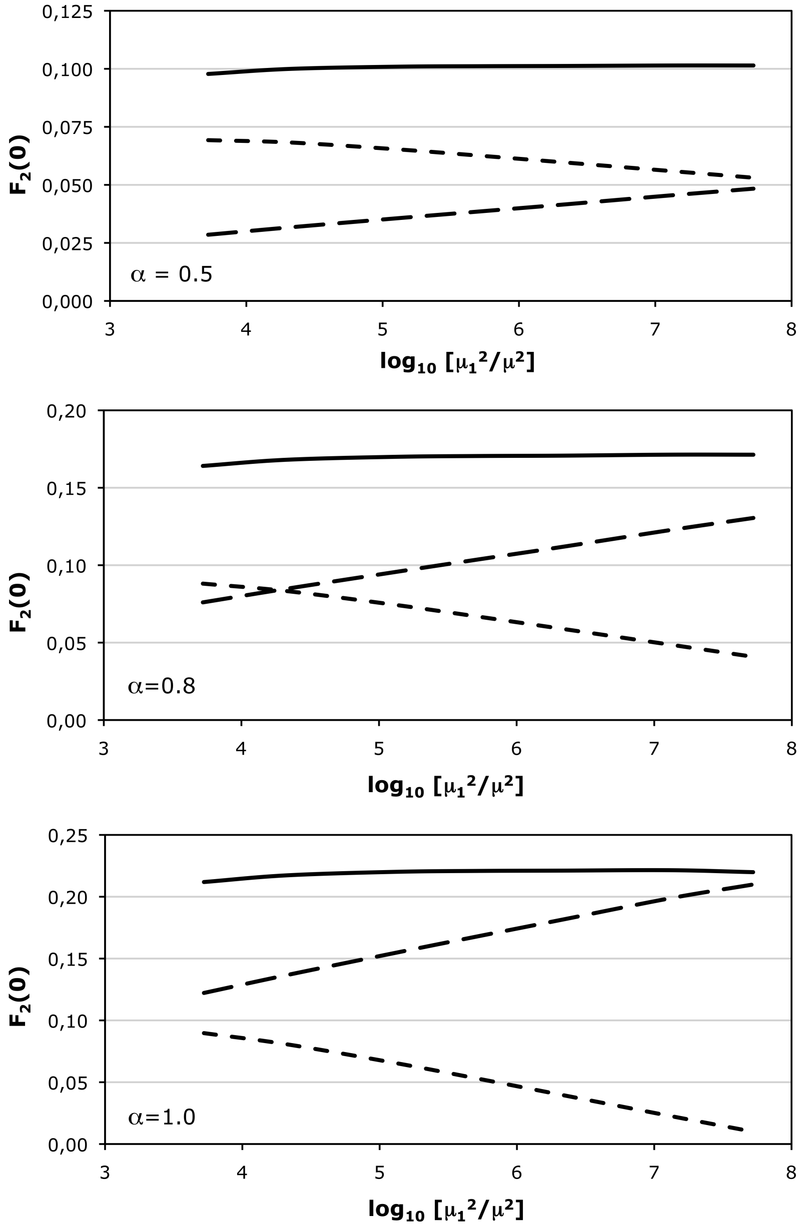

Figure 4: The anomalous magnetic moment in the Yukawa model as a function of the Pauli–Villars mass μ_1, for three different values of the coupling constant, $\alpha = 0.5$ (upper plot), 0.8 (middle plot) and $\alpha = 1.0$ (lower plot). The dashed and long-dashed lines are, respectively, the two- and three-body contributions, while the solid line is the total result.

limit $m_1 \to \infty$, we continue to work with the vertex functions and the electromagnetic vertex taken in this limit. The independence of physical results of the order in which the infinite Pauli–Villars mass limit is taken and, hence, on the way we use to get rid of the bare parameters, is a strong evidence of the self-consistency of our renormalization scheme.

The AMM of electron was calculated non-perturbatively, in the $N = 2$ truncation and with the oscillator basis, in Ref. [13] (see Ref. [14] for the review).

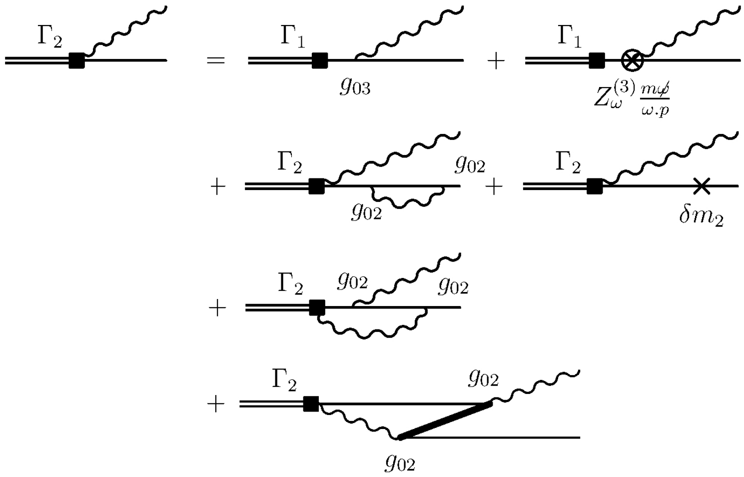

Figure 5: Graphical representation of the equation for the two-body vertex function including the contribution of antifermion d.o.f. in the quenched approximation.

4 Antifermion degrees of freedom

We extend the Fock decomposition of the fermion state vector by introducing the antifermion d.o.f. In the lowest (also three-body) approximation this corresponds to adding the $ff\bar{f}$ Fock sector to those previously introduced (f, fb, and fbb). In the three-body approximation, this new Fock component is easily expressed through the two-body component, as the fbb one. As a result, we obtain a closed (matrix) equation for the two-body vertex function, as given, in the quenched approximation, by Fig. 5. It differs from the equation in the $f + fb + fbb$ approximation, shown in Fig. 2, by an additional term on the right-hand side (the last diagram in Fig. 5).

It turns out that the antifermion contribution makes a week influence on values of form factors, but these values are now obtained with the parameters $g_{03}(x), Z_\omega(x)$ which are much more flat functions of x than without the antifermion.

In Figs. 6 and 7 these bare parameters are shown as a function of x, each for $\alpha = 0.5$, 0.8, and 1.0, at a typical value $\mu_1 = 100$. In Fig. 6 the relative value of g'_{03} with respect to its mean value $\bar{g}'_{03}$ over the interval $0 \le x \le 1$ is shown, i. e. we plot the quantity

$$\delta g'_{03}(x) = [g'_{03}(x) - \bar{g}'_{03}]/\bar{g}'_{03},$$

where $\bar{g}'_{03} = \int_0^1 g'_{03}(x)\,dx$. The "prime" indicates that $g'_{03}(x)$ and $Z'_\omega(x)$ include some factors precised in Ref. [5]. For comparison, we show also in these plots the same functions calculated without antifermion contributions. The most interesting fact is that the function $g'_{03}(x)$, which exhibits strong x-dependence in the $f + fb + fbb$ approximation, becomes almost a constant if the $ff\bar{f}$ Fock sector is included. Concerning the function $Z'_\omega(x)$, it shows a similar tendency as well, with a bit stronger x-dependency than $g'_{03}(x)$. In addition, the magnitude of $Z'_\omega(x)$ is reasonably smaller than that calculated in the $f + fb + fbb$ truncated Fock space.

5 Conclusion

We have developed a non-perturbative approach to field theory based on the Fock decomposition and its truncation. It includes also the non-perturbative renormalization. The approach is applied to calculations of electromagnetic form factors in the Yukawa model. Truncating the Fock space up to the three-body sector fbb and then including $ff\bar{f}$, we calculated anomalous magnetic moment of fermion. It is rather

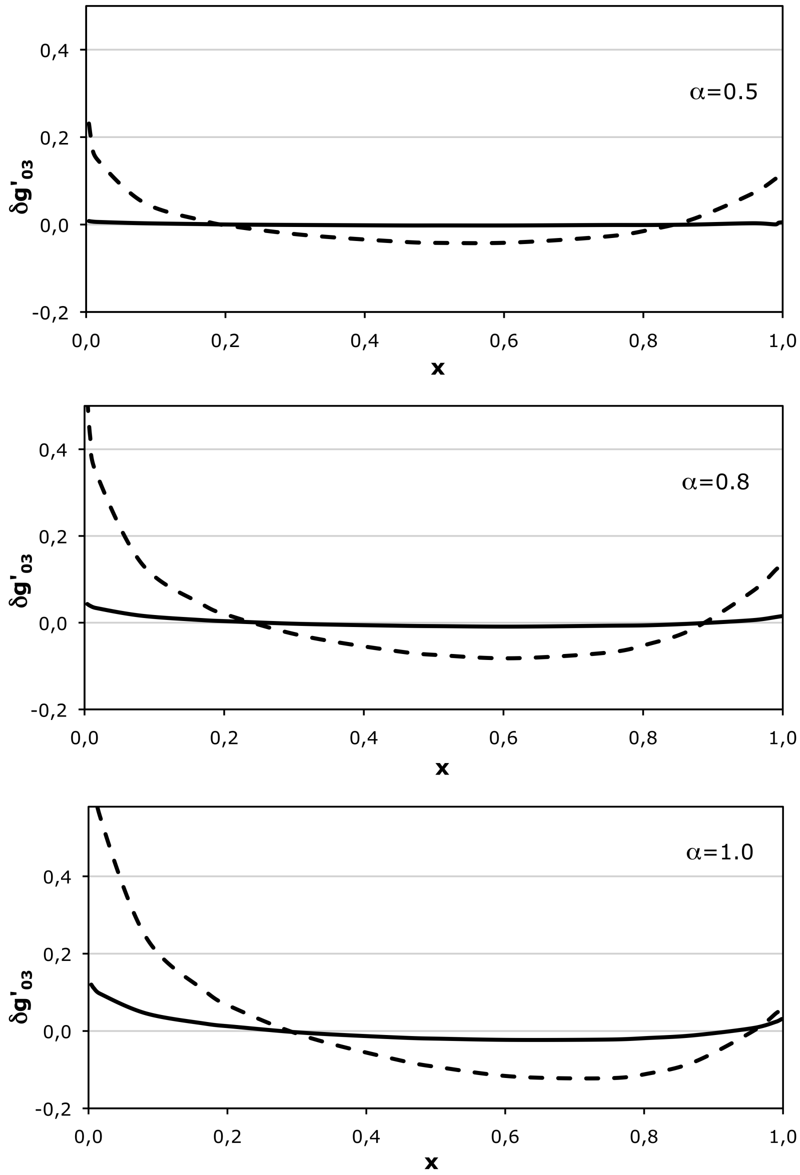

Figure 6: x-dependence of the bare coupling constant g'_{03} calculated for $\mu_1 = 100$ relatively to its mean value over the interval $x \in [0, 1]$ for $\alpha = 0.5$ (upper plot), $\alpha = 0.8$ (middle plot) and $\alpha = 1.0$ (lower plot). The solid (dashed) lines correspond to the results obtained with (without) the $ff\bar{f}$ Fock sector contribution.

stable relative to the increase of the regulator — the Pauli–Villars meson mass μ_1, that indicates that in this way we find the convergent results. Due to the truncation, the bare parameters in the truncated Hamiltonian depend on the kinematical variable x. This dependence becomes weaker when the $ff\bar{f}$ sector is included.

The numerical results for the $N = 3$ truncation presented above were obtained by a laptop. In order to go further, one should certainly use supercomputers which open wide perspectives for the non-perturbative calculations in the field theory. This can make an alternative to the lattice calculations. For a review of this field, applications to the light-front Hamiltonian dynamics and the results of *ab initio* calculations in nuclear physics see Refs. [14, 15]. It would be very important, as the next step, to carry out calculations for the four-body truncation (to solve the equations shown in

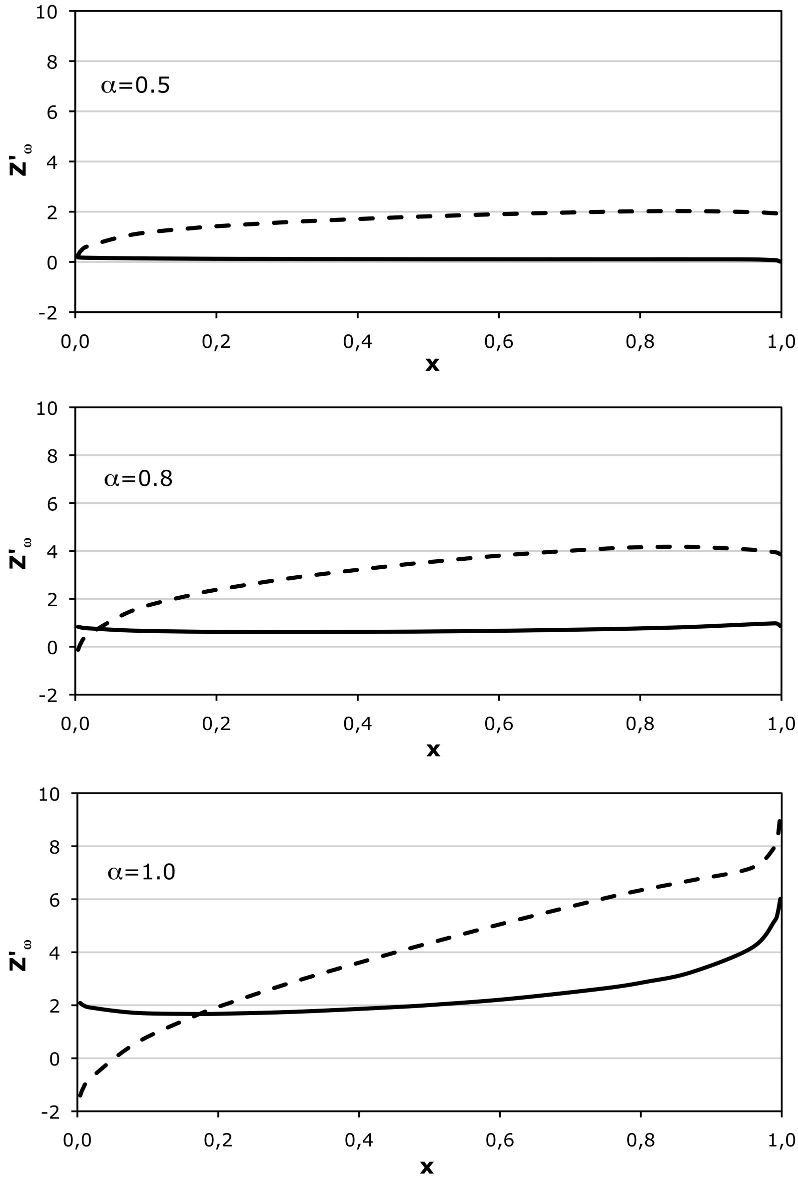

Figure 7: The same as Fig. 6 but for the x-dependence of the counterterm Z'_ω.

Fig. 3) in order to check a possible convergence relative to the number of incorporated Fock sectors. From the Yukawa model which serves as a testing area for development of methods, one should go over to a realistic field theory.

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