Structure of T-Matrices on Unphysical Energy Sheets and Few-Body Resonances*

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**International Conference "Nuclear Theory in the Supercomputing Era – 2016" (NTSE-2016), September 19–23, 2016, Pacific National University, Khabarovsk, Russia*

Contents

1 Introduction

With a resonance one usually associates an unstable (metastable) state that only exists during a certain time.

S-matrix interpretation

Gamov (1928): resonances $\Longleftrightarrow$ poles of the scattering amplitudes
[α decay of heavy nuclei] (that is, those of the S-matrix)

Titchmarsh (1946): Resonances are also poles of the continued resolvent kernel

Jost functions (1940's)

***Complex scaling approach* (rotation of the continuous spectrum)**

Lovelace ($\sim$ 1964), Balslev and Combes (1971) . . .

Hagedorn (1979): for a wide class of potentials the scaling resonances are also the scattering matrix resonances.

We also mention the ***Lax-Phillips approach*** and various versions of ***perturbation theory for resonances*** (Albeverio, Livšic, Howland, Rauch, ...)

In contrast to the "normal" bound and scattering states, from the mathematical point of view the resonant ones are still a mysterious subject. Many questions still remain unanswered.

For example: How to describe ("accurately" from the mathematical point of view) scattering of a particle off a resonant state of other two particles?

There is a difficulty even with the definition of resonance:

> Resonances are NOT a unitary invariant of a self-adjoint (Hermitian) operator

J. S. Howland (1974), B. Simon (1978): ***No satisfactory definition of resonance can rely on a single operator on an abstract Hilbert space***. Always an extra structure is necessary. Say, an unperturbed dynamics (in quantum scattering theory) or geometric setup (in acoustical or optical problems)..

> ***Resonances are always as relative as the scattering matrix itself***.

We follow the **typical setup**

Kinetic energy operator $H_0 \Longleftrightarrow$ unperturbed dynamics

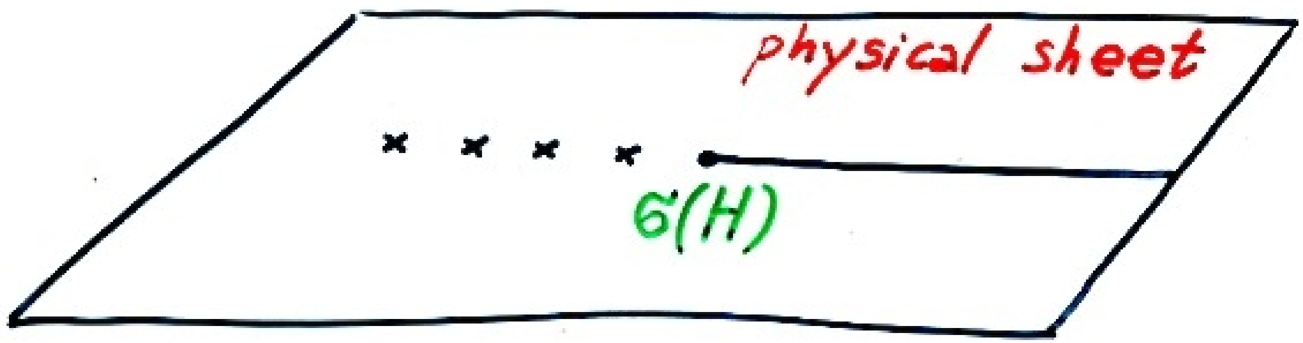
$$H = H_0 + V, \qquad V \quad \text{interaction.}$$

The resolvent

$$R(z) = (H - z)^{-1}$$

is an analytic operator-valued function of $z \in \mathbb{C} \setminus \sigma(H)$.

The spectrum $\sigma(H)$ is a natural boundary for holomorphy domain of $R(z)$ considered as an operator-valued function.



However the kernel $R(\cdot,\cdot,z)$ (given in some specific representation) may admit analytic continuation through the continuous spectrum $\sigma_c(H)$.

Or the bilinear form (scalar product) $\langle R(z)\varphi,\psi\rangle$ admits such a continuation for any φ,ψ of a dense subset of the Hilbert space $\mathscr{H}$.

Or the "augmented" resolvent $PR(z)P$ admits analytic continuation for P the orthogonal projection onto a subspace of $\mathscr{H}$.

In any case one deals with the Riemann surface of a holomorphic function.

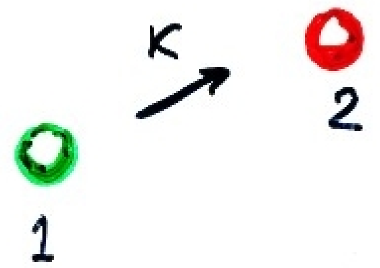
Theorem on the uniqueness of analytic continuation: *Analytic function is uniquely defined by its values on an infinite set in $\mathbb{C}$ having limiting point(s)*. Thus, if one knows the resolvent $R(z)$ (or T-matrix, S-matrix) on the physical sheet then one may, in principle, to express it on unphysical sheets through its values in the physical sheet.

Some time ago, we have derived just such expressions: ***Explicit representations for $R(z)$, $T(z)$, and $S(z)$ on unphysical sheets in terms of these quantities themselves taken from the physical sheet.***

In particular, ***these representations show which blocks of the scattering matrix (taken on the physical sheet) are "responsible" for resonances on a certain unphysical sheet.*** (In such a case all the study of resonances would reduce to a work completely on the physical sheet!)

These blocks of the scattering matrix are some its "truncations". A ***resonance*** on a certain unphysical sheet is nothing but the (complex) ***energy*** at which the corresponding ***truncated scattering matrix has eigenvalue zero***.

2 Two-Body Problem
in the center-of-mass frame



$$\boldsymbol{k} = \left[\frac{\mathrm{m}_1 + \mathrm{m}_2}{2\mathrm{m}_1\mathrm{m}_2}\right]^{1/2} \cdot \frac{\mathrm{m}_1\mathbf{p}_2 - \mathrm{m}_2\mathbf{p}_1}{\mathrm{m}_1 + \mathrm{m}_2} \qquad \text{(reduced relative momentum)}$$

$$(hf)(\boldsymbol{k}) = \boldsymbol{k}^2 f(\boldsymbol{k}) + (Vf)(\boldsymbol{k})$$

$V(\boldsymbol{k}, \boldsymbol{k}') = V(\boldsymbol{k} - \boldsymbol{k}')$ — in case of local potentials —
and $V(\boldsymbol{k}) = \overline{V(-\boldsymbol{k})}$, $\boldsymbol{k} \in \mathbb{R}^3$.

For simplicity we assume that $V(\boldsymbol{k})$ is holomorphic for all $\boldsymbol{k} \in \mathbb{C}^3$. Such a situation takes place if, in coordinate representation, say, $V \in C^\infty(\mathbb{R}^3)$ and has a compact support.

(In fact it suffices to require the holomorphy of $V(\boldsymbol{k})$ only in a "strip" $|\mathrm{Im}\,\boldsymbol{k}| < a$ for some $a > 0$.)

Resolvents (Green functions):

$$r_0(z) = (h_0 - z)^{-1}, \qquad (h_0 f)(\boldsymbol{k}) = \boldsymbol{k}^2 f(\boldsymbol{k}),$$
$$r(z) = (h - z)^{-1}.$$

$$r_0(\boldsymbol{k}, \boldsymbol{k}', z) = \frac{\delta(\boldsymbol{k} - \boldsymbol{k}')}{\boldsymbol{k}^2 - z}$$

T-operator (T-matrix):

$$t(z) = V - V r(z) V \quad \Longrightarrow \quad r(z) = r_0(z) - r_0(z) t(z) r_0(z)$$

The Lippmann-Schwinger equation for $t(z)$

$$t(z) = V - V r_0(z) t(z),$$

that is

$$t(\boldsymbol{k},\boldsymbol{k}',z) = V(\boldsymbol{k},\boldsymbol{k}') - \int\limits_{\mathbb{R}^3} d\boldsymbol{q} \frac{V(\boldsymbol{k},\boldsymbol{q})t(\boldsymbol{q},\boldsymbol{k}',z)}{\boldsymbol{q}^2 - z}. \tag{2.1}$$

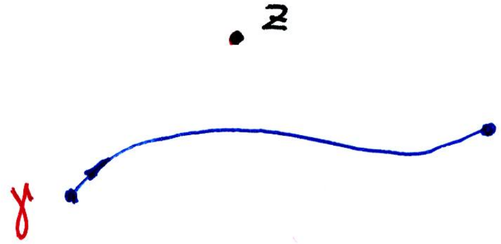
Clearly, all the dependence of t on z in (2.1) is determined by the integral term on the r.h.s. part. This integral is nothing but a particular case of the Cauchy type integral

$$\Phi(z) = \int\limits_{\mathbb{R}^N} d\boldsymbol{q} \frac{f(\boldsymbol{q})}{\lambda + \boldsymbol{q}^2 - z}$$ (here, $N = 3$; $N = 6$ in the Faddeev eqs.).

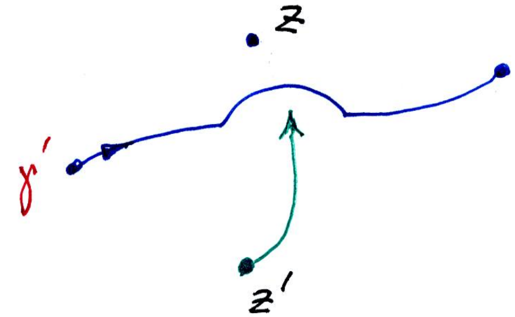
Cauchy-type integrals of the same form are also present in the three-body Faddeev equations.

Analytic continuation of the Cauchy integral

$$F(z) = \int_\gamma d\xi \, \frac{f(\xi)}{\xi - z}$$

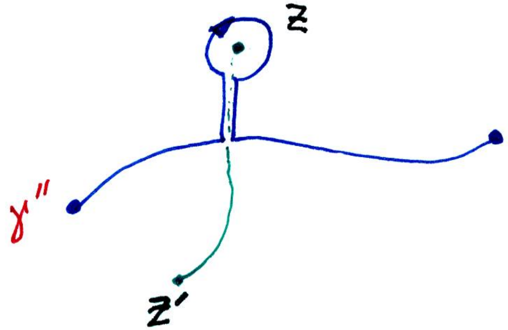


with a holomorphic f through the integration path γ. (f is assumed to be holomorphic in a domain $\mathscr{D} \subset \mathbb{C}$ having a non-empty intersection with γ.)



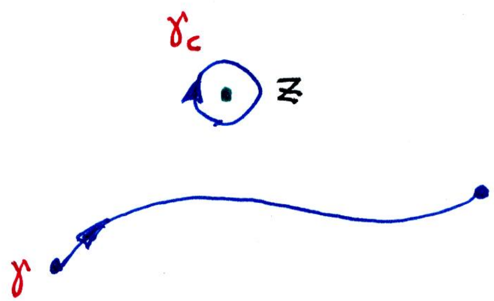
After continuation across γ from the bottom up,

$$\begin{aligned} \widetilde{F}(z) &= \int_\gamma d\xi \, \frac{f(\xi)}{\xi - z} - 2\pi \mathrm{i}\, f(z) \\ &= F(z) - 2\pi \mathrm{i}\, f(z). \end{aligned}$$



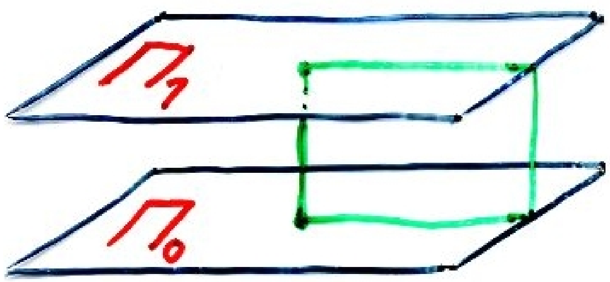
Similarly, after continuation across γ from the top down,

$$\begin{aligned} \widetilde{F}(z) &= \int_\gamma d\xi \, \frac{f(\xi)}{\xi - z} + 2\pi \mathrm{i}\, f(z) \\ &= F(z) + 2\pi \mathrm{i}\, f(z). \end{aligned}$$



Denote by $\mathfrak{R}_\lambda$ the Riemann surface of the function

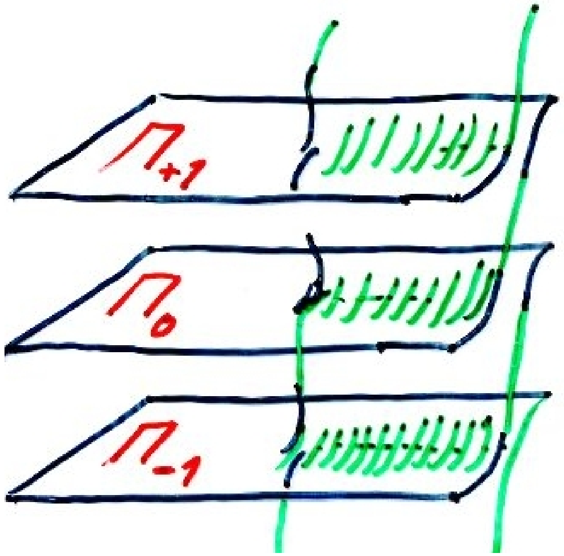
$$\zeta(z) = \begin{cases} (z-\lambda)^{1/2}, & N \quad \text{odd}, \\ \ln(z), & N \quad \text{even}. \end{cases}$$



$\ell = 1 \quad (z-\lambda)^{1/2} = -\sqrt{z-\lambda}$

$\ell = 0 \quad (z-\lambda)^{1/2} = \sqrt{z-\lambda}$

N odd



$$\left.\begin{array}{l} \ell = +1 \\ \ell = 0 \\ \ell = -1 \end{array}\right\} \ln(z-\lambda) = \ln|z-\lambda| + i\varphi + 2\pi i \ell$$

N even

Lemma 1. *For a holomorphic $f(\boldsymbol{q})$, $\boldsymbol{q} \in \mathbb{C}^N$, the function*

$$\Phi(z) = \int_{\mathbb{R}^N} d\boldsymbol{q} \frac{f(\boldsymbol{q})}{\lambda + \boldsymbol{q}^2 - z}$$

is holomorphic on $\mathbb{C} \setminus [\lambda, +\infty)$ and admits the analytic continuation onto $\mathfrak{R}_\lambda$ as follows

$$\Phi(z)|_{\Pi_l} = \Phi(z) - l\,\pi \mathrm{i}(\sqrt{z-\lambda})^{N-2} \int_{S^{N-1}} d\widehat{\boldsymbol{q}} f(\sqrt{z-\lambda}\,\widehat{\boldsymbol{q}}). \qquad \textbf{(2.2)}$$

Do not confuse l with orbital moment! Here l is the first letter of the Russian word "*list*" $\Leftrightarrow$ "*sheet*".

Now for $r_0(z) = (h_0 - z)^{-1}$ set

$$\big(r_0(z)f_1, f_2\big) \equiv \int_{\mathbb{R}^3} d\boldsymbol{q} \frac{f_1(\boldsymbol{q})f_2(\boldsymbol{q})}{\boldsymbol{q}^2 - z} \qquad \big(= \Phi(z), \quad \lambda = 0\big),$$

where f_1 and f_2 are holomorphic functions. Then by Lemma 1

$$\big(r_0(z)f_1, f_2\big)|_{\Pi_1} = \big(r_0(z)f_1, f_2\big)|_{\Pi_0} - \pi \mathrm{i}\sqrt{z} \int_{S^2} d\widehat{\boldsymbol{q}} f_1(\sqrt{z}\widehat{\boldsymbol{q}}) f_2(\sqrt{z}\widehat{\boldsymbol{q}})$$
$$\Updownarrow$$
$$r_0(z)|_{\Pi_1} = r_0(z) + \mathrm{a}_0(z)\mathrm{j}^\dagger(z)\mathrm{j}(z),$$

where $\mathrm{a}_0(z) = -\pi\mathrm{i}\sqrt{z}$, and

$$\big(\mathrm{j}(z)f\big)(\widehat{\boldsymbol{k}}) = f(\sqrt{z}\widehat{\boldsymbol{k}}).$$

What can be said about the operator $t(z)$?

$$t(z) = V - V r_0 t.$$

After the continuation to the sheet Π_1 we obtain

$$t' = V - V(r_0 + \mathrm{a}_0 \mathrm{j}^\dagger \mathrm{j})t', \qquad t' = t|_{\Pi_1},$$

which implies that

$$(I + V r_0)t' = V - \mathrm{a}_0 V \mathrm{j}^\dagger \mathrm{j}\, t'.$$

Perform inversion of $(I + V r_0)$ taking into account that $t(z) = V - V r_0 t$ and, hence, $(I + V r_0)^{-1} V = t$:

$$t' = t - \mathrm{a}_0\, t \mathrm{j}^\dagger \mathrm{j}\, t'. \tag{2.3}$$

Further on, apply j to the both parts and get

$$\mathrm{j}\,t' = \mathrm{j}\,t - \mathrm{a}_0 \mathrm{j}\,t \mathrm{j}^\dagger\, \mathrm{j}\,t',$$

which means

$$(I + \mathrm{a}_0 \mathrm{j}\,t \mathrm{j}^\dagger)\mathrm{j}\,t' = \mathrm{j}\,t. \tag{2.4}$$

Notice that

$$\boxed{I + \mathrm{a}_0 \mathrm{j}\,t \mathrm{j}^\dagger = s(z)} \qquad \text{is the scattering matrix,}$$

$$s(\widehat{\boldsymbol{k}}, \widehat{\boldsymbol{k}}', z) = \boldsymbol{\delta}(\widehat{\boldsymbol{k}}, \widehat{\boldsymbol{k}}') - \boldsymbol{\pi}\mathrm{i}\sqrt{z}\, t(\sqrt{z}\widehat{\boldsymbol{k}}, \sqrt{z}\widehat{\boldsymbol{k}}', z).$$

Hence,

$$\mathrm{j}\,t' = [s(z)]^{-1}\mathrm{j}\,t$$

Come back to Eq. (2.3) and conclude that

$$\boxed{t' = t - \mathrm{a}_0\, t \mathrm{j}^\dagger \left[s(z)\right]^{-1} \mathrm{j}\, t,} \tag{2.5}$$

$$\mathrm{a}_0(z) = -\pi \mathrm{i}\sqrt{z},$$

that is

$$\boxed{t(z)|_{\Pi_1} = t(z) - \mathrm{a}_0(z)\, t(z) \mathrm{j}^\dagger(z) \left[s(z)\right]^{-1} \mathrm{j}(z)\, t(z).} \tag{2.6}$$

All the entries on the r.h.s. of (2.6) are taken on the physical sheet!

From (2.6) we derive that

$$\boxed{s(z)|_{\Pi_1} = \mathscr{E}\,[s(z)]^{-1}\,\mathscr{E},}\,,$$

where $\mathscr{E}$ is the inversion, $(\mathscr{E}f)(\widehat{\boldsymbol{k}}) = f(-\widehat{\boldsymbol{k}})$.

In a similar way,

$$\boxed{r(z)|_{\Pi_1} = r + \mathrm{a}_0\,(I - rV)\mathrm{j}^{\dagger}\,[s(z)]^{-1}\mathrm{j}(I - Vr).}$$

Hence the resonances are nothing but zeros of $s(z)$ in the physical sheet, that is,

$$\boxed{z \text{ is a resonance} \iff \text{there is } \mathscr{A} \in L_2(S^2) \text{ such that } s(z)\mathscr{A} = 0.}$$

3 Multichannel problem with binary channels

$$h = \begin{pmatrix} \lambda_1 + h_0^{(1)} + V_{11} & V_{12} & \dots & V_{1m} \\ V_{21} & \lambda_2 + h_0^{(2)} + V_{22} & \dots & V_{2m} \\ \dots & \dots & \dots & \dots \\ V_{m1} & V_{m2} & \dots & \lambda_m + h_0^{(m)} + V_{mm} \end{pmatrix},$$

in momentum representation

$$(h_0^{(\alpha)} f_\alpha)(\boldsymbol{k}_\alpha) = \boldsymbol{k}_\alpha^2 f_\alpha(\boldsymbol{k}_\alpha), \quad \boldsymbol{k}_\alpha \in \mathbb{R}^{n_\alpha}, \quad f_\alpha \in L_2(\mathbb{R}^{n_\alpha}), \quad \alpha = 1, 2, \dots, m.$$

Assume that $V(\boldsymbol{k}_\alpha, \boldsymbol{k}_\beta)$ are analytic in $\boldsymbol{k}_\alpha \in \mathbb{C}^{n_\alpha}, \boldsymbol{k}_\beta \in \mathbb{C}^{n_\beta}$ and sufficiently rapidly decreasing along $\mathrm{Re}\, \boldsymbol{k}_\alpha$ and $\mathrm{Re}\, \boldsymbol{k}'_\beta$

Channel dimensions $n_\alpha \geq 3$.

The thresholds: distinct, $\lambda_1 < \lambda_2 < \dots < \lambda_m$.

Riemann surface $\mathfrak{R}$ coincides with that of the vector-valued function

$$\zeta(z) = \bigl(\zeta_1(z), \zeta_2(z), \ldots, \zeta_m(z)\bigr),$$

where

$$\zeta_\alpha(z) = \begin{cases} (z-\lambda_\alpha)^{1/2} & \text{if } n_\alpha \text{ is odd,} \\ \log(z-\lambda_\alpha) & \text{if } n_\alpha \text{ is even,} \end{cases}$$
$$\alpha = 1, 2, \ldots, m.$$

To enumerate the sheets of $\mathfrak{R}$ it is natural to use a multiindex

$$l = (l_1, l_2, \ldots, l_m),$$

Below we assume that all n_α's are **odd**. (For more sophisticated case of n_α we refer to [AM, Phys. Atom. Nucl. ***77*** (2014), 453].)

Then 2^m sheets and $l_\alpha = 0$ or $l_\alpha = 1$.

Π_l — notation for the sheets of $\mathfrak{R}$.

Result:

$$\boxed{t(z)\big|_{\Pi_l} = t(z) - t(z)\mathrm{J}^\dagger(z) L A(z) [s_l(z)]^{-1} L \mathrm{J}(z) t(z)},$$

where

$$L = \begin{pmatrix} l_1 & 0 & \dots & 0 \\ 0 & l_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & l_m \end{pmatrix}, \quad A(z) = -\pi \mathrm{i} \begin{pmatrix} (\sqrt{z-\lambda_1})^{n_1-2} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & (\sqrt{z-\lambda_m})^{n_m-2} \end{pmatrix},$$

$$\mathrm{J}(z) = \begin{pmatrix} \mathrm{j}_1(z) & 0 & \dots & 0 \\ 0 & \mathrm{j}_2(z) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathrm{j}_m(z) \end{pmatrix}, \quad \big(\mathrm{j}_\alpha(z) f\big)(\widehat{\boldsymbol{q}}) = f(\sqrt{z-\lambda_\alpha}\,\widehat{\boldsymbol{q}}),$$

$$\boxed{s_l(z) = I + L\big(s(z) - I\big)L = I + L\mathrm{J}t(z)\mathrm{J}^\dagger L A(z).}$$

The operator $s_l(z)$ represents the result of ***truncation*** of the total scattering matrix

$$s(z) = I + \mathrm{J}t(z)\mathrm{J}^\dagger A(z).$$

Representations for $t(z)\big|_{\Pi_l} \Longrightarrow$ Explicit representations for $s(z)\big|_{\Pi_l}$ and $r(z)\big|_{\Pi_l}$

Main conclusion:

$$\boxed{z \text{ is a resonance on } \Pi_l \Longleftrightarrow \text{there is } \mathscr{A} \text{ such that } s_l(z)\mathscr{A} = 0.}$$

Underline: $\mathscr{A}_\alpha \neq 0$ only for those channels α where $l_\alpha \neq 0$. This means

$$(I - L)\mathscr{A} = 0.$$

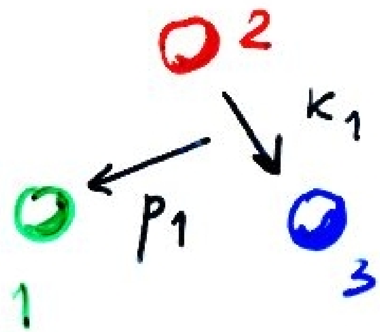
Along with $\mathscr{A}$ we introduce «extended» vector $\widetilde{\mathscr{A}}$ defined by

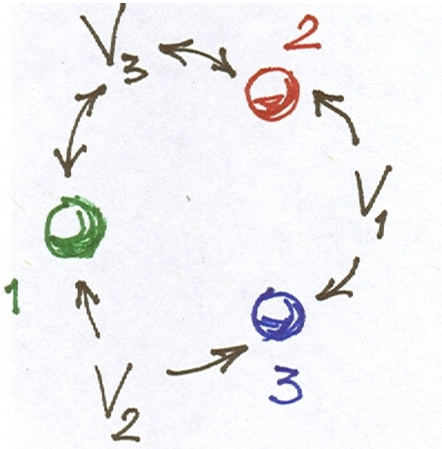
$$\widetilde{\mathscr{A}} = -\mathrm{J}t(z)\mathrm{J}^{\dagger}LA(z)\mathscr{A}\,. \tag{3.1}$$

Obviously, $\mathscr{A} = L\widetilde{\mathscr{A}}$.

Our claim is that up to numerical coefficients, the components $\widetilde{\mathscr{A}_1}(\widehat{k}_1)$, $\widetilde{\mathscr{A}_2}(\widehat{k}_2)$, ..., $\widetilde{\mathscr{A}_m}(\widehat{k}_m)$ of the vector $\widetilde{\mathscr{A}}$ represent the breakup amplitudes for the corresponding resonance state along the channels 1, 2,..., m. Just these amplitudes determine the angular dependence of the asymptotical spherical waves in the channel components of the corresponding ≪Gamow vector≫ $\psi^{\#}_{\mathrm{res}}$ in the coordinate representation.

4 Three-Body Problem





$$\boldsymbol{k}_\alpha = \left(\frac{\mathrm{m}_\beta + \mathrm{m}_\gamma}{2\mathrm{m}_\beta \mathrm{m}_\gamma}\right)^{1/2} \frac{\mathrm{m}_\beta \mathbf{p}_\gamma - \mathrm{m}_\gamma \mathbf{p}_\beta}{\mathrm{m}_\beta + \mathrm{m}_\gamma},$$

$$\boldsymbol{p}_\alpha = \left(\frac{\mathrm{m}_1 + \mathrm{m}_2 + \mathrm{m}_3}{2\mathrm{m}_\alpha(\mathrm{m}_\beta + \mathrm{m}_\gamma)}\right)^{1/2} \frac{(\mathrm{m}_\beta + \mathrm{m}_\gamma)\mathbf{p}_\alpha - \mathrm{m}_\alpha(\mathbf{p}_\beta + \mathbf{p}_\gamma)}{\mathrm{m}_1 + \mathrm{m}_2 + \mathrm{m}_3}$$

Here, (α, β, γ) is a cyclic permutation of the indices $(1, 2, 3)$.

$$H_0 = \boldsymbol{k}_\alpha^2 + \boldsymbol{p}_\alpha^2, \qquad V = V_1 + V_2 + V_3, \qquad H = H_0 + V$$

$$R_0(z) = (H_0 - z)^{-1}, \qquad R(z) = (H - z)^{-1}$$

T-operator: $\quad T(z) = V - VR(z)V$

Faddeev components:

$$M_{\alpha\beta} = \delta_{\alpha\beta}V_\alpha - V_\alpha R(z)V_\beta \qquad (\alpha,\beta = 1,2,3)$$

Faddeev equations in the matrix form:

$$\boxed{M(z) = \mathbf{t}(z) - \mathbf{t}(z)\mathbf{R}_0(z)\Upsilon M(z)\,.}$$

Here

$$\mathbf{R}_0 = \begin{pmatrix} R_0 & 0 & 0 \\ 0 & R_0 & 0 \\ 0 & 0 & R_0 \end{pmatrix} \quad \text{and} \quad \mathbf{t} = \begin{pmatrix} \mathbf{t}_1 & 0 & 0 \\ 0 & \mathbf{t}_2 & 0 \\ 0 & 0 & \mathbf{t}_3 \end{pmatrix}$$

with

$$\boxed{\mathbf{t}_\alpha(P,P',z) = t_\alpha(\boldsymbol{k}_\alpha,\boldsymbol{k}'_\alpha,z-\boldsymbol{p}_\alpha^2)\delta(\boldsymbol{p}_\alpha-\boldsymbol{p}'_\alpha).}$$

$$\Upsilon = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$$

Let h_α be the Hamiltonian of the two-body subsystem α, and ε_α and ψ_α the (only) binding energy and b.s. wave function, respectively, that is,

$$h_\alpha \psi_\alpha = \varepsilon_\alpha \psi_\alpha.$$

Then

$$t_\alpha(\boldsymbol{k},\boldsymbol{k}',z) = -\frac{\varphi_\alpha(\boldsymbol{k})\overline{\varphi_\alpha(\boldsymbol{k}')}}{\varepsilon_\alpha - z} + \widetilde{t}_\alpha(\boldsymbol{k},\boldsymbol{k}',z)$$

with the formfactor

$$\varphi_\alpha = V_\alpha \psi_\alpha.$$

Recall that

$$R_0(P,P',z) = \frac{\delta(P-P')}{P^2-z}.$$

These kernels (associated with the corresponding thresholds) are the sources of the Cauchy type integrals in Faddeev equations.

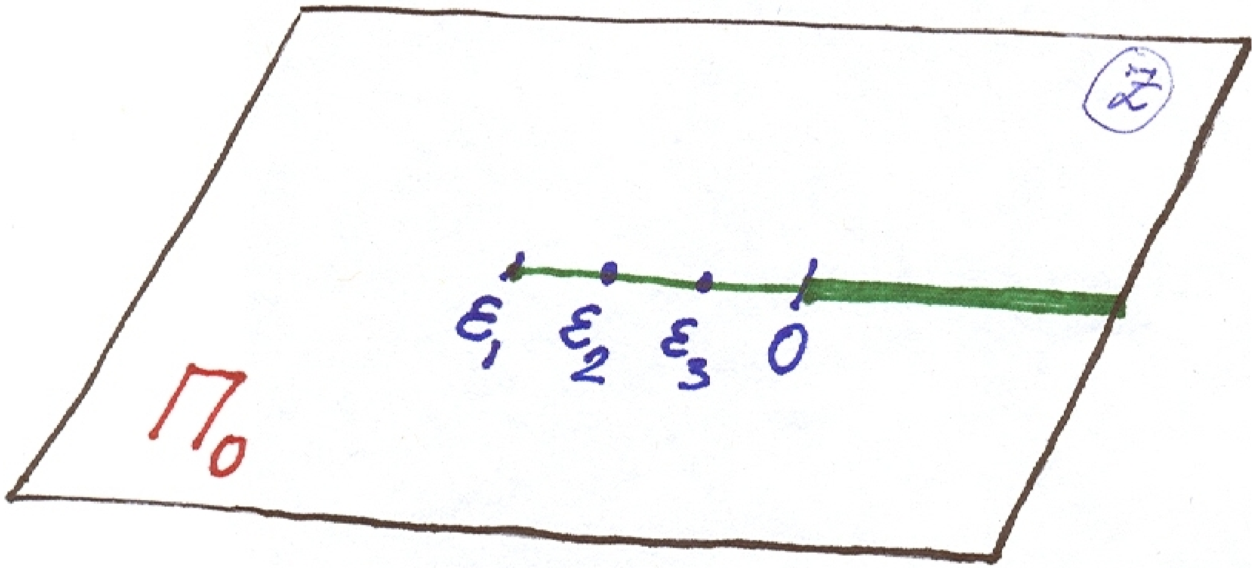
Further, we perform the analytic continuation of the Faddeev equations. It is remarkable that

the continued Faddeev equations can be solved explicitly (!)
— in terms of the matrix M itself,

and the "values" of $M(z)$ are taken exclusively from the physical energy sheet. The situation is very the same as in the case of the two-body T-matrix.

Surely, the result of continuation depends on the unphysical sheet under consideration.

How many sheets do we have in the three-body case?



Two-body binding energies ε_1, ε_2, ε_3 are square root branching points
The three-body threshold 0 is a logarithmic branching point

Hence, only encircling the two-body thresholds one arrives at **seven** different unphysical sheets.

The three-body threshold generates infinitely many unphysical sheets.

There is also a "fine structure": in particular, additional branching points, already on the unphysical sheets, may be generated by the two-body resonances. We did not yet have a look at the unphysical sheets of the "second order".

In order to enumerate the sheets (of the "first" order only) we need a multi-index,

$$l = (l_0, l_1, l_2, l_3),$$

with

$$\begin{aligned} l_0 &= \ldots, -1, 0, 1, \ldots \qquad \text{(0 physical, } \pm 1, \pm 2, \ldots \text{ unphysical)} \\ l_\alpha &= 0, 1 \qquad \text{(0 physical, 1 unphysical)} \end{aligned}$$

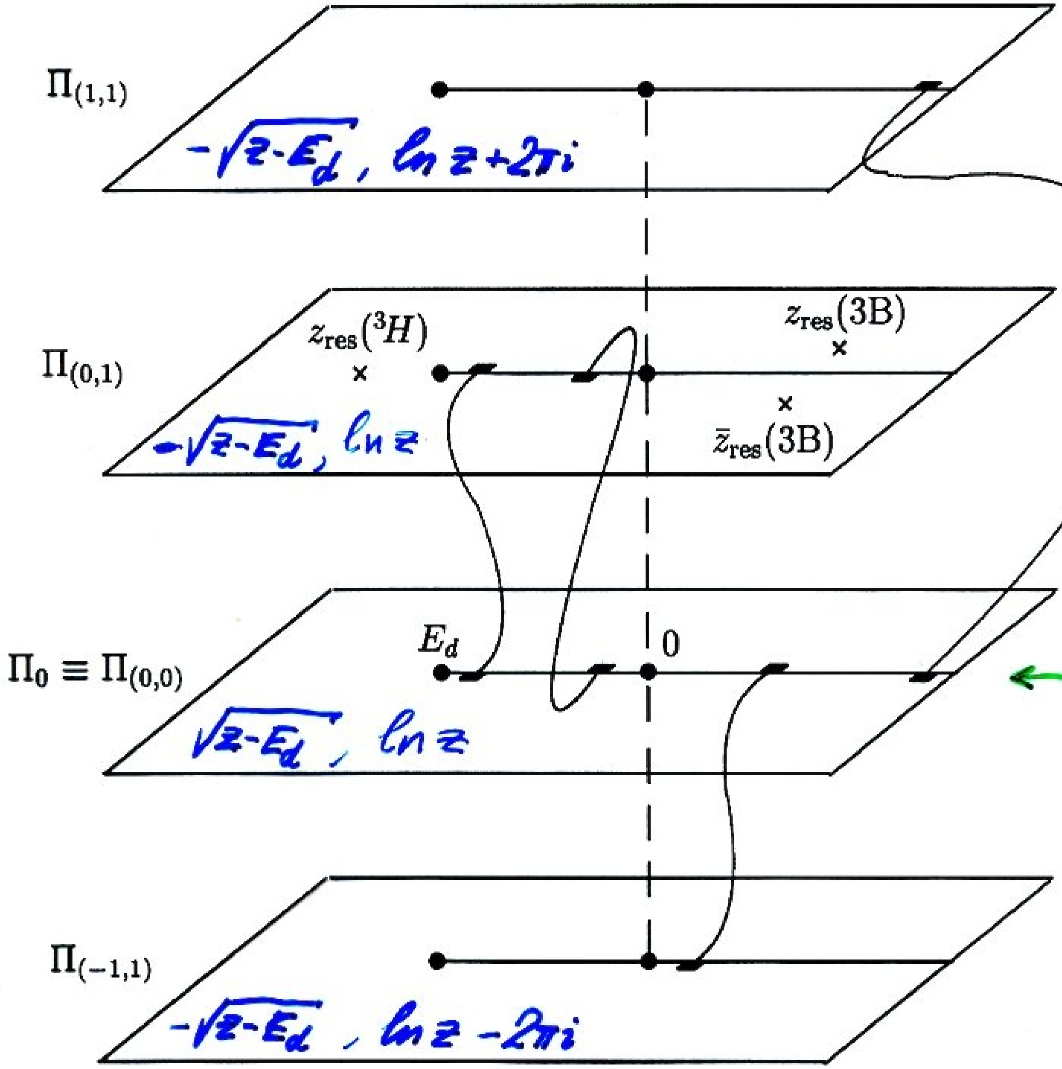
$$\Pi_l \quad \text{the corresponding unphysical sheet}$$

Also introduce

$$L = \begin{pmatrix} l_0 & 0 & 0 & 0 \\ 0 & l_1 & 0 & 0 \\ 0 & 0 & l_2 & 0 \\ 0 & 0 & 0 & l_3 \end{pmatrix} \qquad \text{and} \qquad \widetilde{L} = \begin{pmatrix} |l_0| & 0 & 0 & 0 \\ 0 & l_1 & 0 & 0 \\ 0 & 0 & l_2 & 0 \\ 0 & 0 & 0 & l_3 \end{pmatrix}.$$

nnp, 3B, $^4He_3$ trimer

$$f(z) = \begin{pmatrix} \ln z \\ (z-E_d)^{1/2} \end{pmatrix}$$



$\leftarrow$ physical sheet

$\Pi_{(\ell_0,\ell_1)}$

Only physical sheet and **three neighboring** sheets of **infinitely many** unphysical sheets are shown here: the only two-cluster unphysical sheet and two three-body

In the simple four-channel case under consideration the three-body scattering matrix is a 4×4 operator matrix of the form

$$S(z)=\widehat{I}+A(z)\widehat{\mathscr{T}}(z),$$

$$\widehat{\mathscr{T}}=\begin{pmatrix} \widehat{\mathscr{T}}_{00} & \widehat{\mathscr{T}}_{01} & \widehat{\mathscr{T}}_{02} & \widehat{\mathscr{T}}_{03} \\ \widehat{\mathscr{T}}_{10} & \widehat{\mathscr{T}}_{11} & \widehat{\mathscr{T}}_{12} & \widehat{\mathscr{T}}_{13} \\ \widehat{\mathscr{T}}_{20} & \widehat{\mathscr{T}}_{21} & \widehat{\mathscr{T}}_{22} & \widehat{\mathscr{T}}_{23} \\ \widehat{\mathscr{T}}_{30} & \widehat{\mathscr{T}}_{31} & \widehat{\mathscr{T}}_{32} & \widehat{\mathscr{T}}_{33} \end{pmatrix},$$

where

$$A(z)=\mathrm{diag}\{-\pi \mathrm{i} z^2, -\pi\mathrm{i}\sqrt{z-\varepsilon_1}, -\pi\mathrm{i}\sqrt{z-\varepsilon_2}, -\pi\mathrm{i}\sqrt{z-\varepsilon_3}\}$$

Up to a scalar function of z the kernel of the entry $\widehat{\mathscr{T}}_{\alpha\beta}$ coincides with the amplitude for the corresponding process,

$$\begin{aligned}
\widehat{\mathscr{T}}_{00}: &\quad 3\longrightarrow 3 \\
\widehat{\mathscr{T}}_{\alpha 0}: &\quad 2\longrightarrow 3, \quad \alpha=1,2,3 \\
\widehat{\mathscr{T}}_{0\beta}: &\quad 3\longrightarrow 2, \quad \beta=1,2,3 \\
\widehat{\mathscr{T}}_{\alpha\beta}: &\quad 2\longrightarrow 2, \quad \alpha,\beta=1,2,3
\end{aligned}$$

The matrix $\widehat{\mathscr{T}}$ is obtained from the matrix

$$\mathscr{T} = \begin{pmatrix} \mathscr{T}_{00} & \mathscr{T}_{01} & \mathscr{T}_{02} & \mathscr{T}_{03} \\ \mathscr{T}_{10} & \mathscr{T}_{11} & \mathscr{T}_{12} & \mathscr{T}_{13} \\ \mathscr{T}_{20} & \mathscr{T}_{21} & \mathscr{T}_{22} & \mathscr{T}_{23} \\ \mathscr{T}_{30} & \mathscr{T}_{31} & \mathscr{T}_{32} & \mathscr{T}_{33} \end{pmatrix}$$

with elements

$$\mathscr{T}_{00}(z) = T(z) = V - VR(z)V, \qquad \Big| \quad \mathscr{T}_{00}(P,P',z)$$

$$\mathscr{T}_{0\beta}(z) = \big(\overline{V}_{\beta} - VR(z)\overline{V}_{\beta}\big)|\psi_{\beta}\rangle, \qquad \Big| \quad \mathscr{T}_{0\beta}(P,p'_{\beta},z)$$

$$\mathscr{T}_{\alpha 0}(z) = \langle\psi_{\alpha}|\big(\overline{V}_{\alpha} - \overline{V}_{\alpha}R(z)V\big), \qquad \Big| \quad \mathscr{T}_{\alpha 0}(p_{\alpha},P',z)$$

$$\mathscr{T}_{\alpha\beta}(z) = \langle\psi_{\alpha}|\big(\overline{V}_{\alpha} - \overline{V}_{\alpha}R(z)\overline{V}_{\beta}\big)|\psi_{\beta}\rangle, \qquad \Big| \quad \mathscr{T}_{\alpha\beta}(p_{\alpha},p'_{\beta},z)$$

$$\alpha,\beta = 1,2,3.$$

(!!) $U_{00} = T$, $U_{0\beta} = \overline{V}_{\beta} - VR\overline{V}_{\beta}$, $U_{\alpha 0} = \overline{V}_{\alpha} - \overline{V}_{\alpha}RV$, $U_{\alpha\beta} = \overline{V}_{\alpha} - \overline{V}_{\alpha}R(z)\overline{V}_{\beta}$

— transition operators

$$\widehat{\mathscr{T}}_{00}(\widehat{P},\widehat{P}',z) = \mathscr{T}_{00}(\sqrt{z}\widehat{P},\sqrt{z}\widehat{P}',z),$$

$$\widehat{\mathscr{T}}_{0\beta}(\widehat{P},\widehat{p}'_{\beta},z) = \mathscr{T}_{0\beta}(\sqrt{z}\widehat{P},\sqrt{z-\varepsilon_{\beta}}\widehat{p}'_{\beta},z),$$

$$\widehat{\mathscr{T}}_{\alpha 0}(\widehat{p}_{\alpha},\widehat{P}',z) = \mathscr{T}_{\alpha 0}(\sqrt{z-\varepsilon_{\alpha}}\widehat{p}_{\alpha},\sqrt{z}\widehat{P}',z),$$

$$\widehat{\mathscr{T}}_{\alpha\beta}(\widehat{p}_{\alpha},\widehat{p}'_{\beta},z) = \mathscr{T}_{\alpha 0}(\sqrt{z-\varepsilon_{\alpha}}\widehat{p}_{\alpha},\sqrt{z-\varepsilon_{\beta}}\widehat{p}'_{\beta},z),$$

Explicitly solving the continued Faddeev equations results in the following

$$\boxed{M|_{\Pi_l} = M + Q_M L S_l^{-1} \widetilde{L} \widetilde{Q}_M.}$$

where Q_M and $\widetilde{Q}_M$ are explicitly written in terms of the Faddeev components $M_{\alpha\beta}$ taken immediately from the physical sheet. S_l is a "truncation" of the total three-body scattering matrix S,

$$S_l = I + A(z) L \widehat{\mathscr{T}} \widetilde{L}.$$

Similarly,

$$\boxed{R|_{\Pi_l} = R + Q_R L S_l^{-1} \widetilde{L} \widetilde{Q}_R.}$$

Therefore, the singularities of $M(z)|_{\Pi_l}$ and $S(z)|_{\Pi_l}$ (as well as the ones of $R(z)|_{\Pi_l}$)) are determined by the inverse truncated scattering matrix in $S_l(z)^{-1}$.

$\widetilde{L}$ is nothing but a projection! — An example at the blackboard.

Thus, to find the resonances on the sheet Π_l one should simply look for the zeros of the truncated scattering matrix $S_l(z)$ in the physical sheet, that is, for the points z where $S_l(z)$ has eigenvalue zero:

$$S_l(z)\mathscr{A} = 0.$$

The vector $\mathscr{A}$ will consist of breakup amplitudes of the resonance state into the channels 0, 1, 2, and 3,

$$\mathscr{A} = \begin{pmatrix} \mathscr{A}_0(\widehat{X}) \\ \mathscr{A}_1(\widehat{\boldsymbol{y}}_1) \\ \mathscr{A}_0(\widehat{\boldsymbol{y}}_2) \\ \mathscr{A}_0(\widehat{\boldsymbol{y}}_3) \end{pmatrix} \qquad \text{(in coordinate space).}$$

To this end one can employ any approach that allows to calculate the corresponding truncation of the scattering matrix (surely, only for the energies z in the physical sheet). That is, any approach that allows to calculate the appropriate scattering, rearrangement and breakup amplitudes.

5 Configuration space. Applications

In order to find the amplitudes involved in S_l, one can use in particular the Faddeev differential equations.

We have employed the two-dimensional partial-wave Faddeev equations (arising as the result of a decomposition of the six-dimensional Faddeev equations over bispherical harmonics).

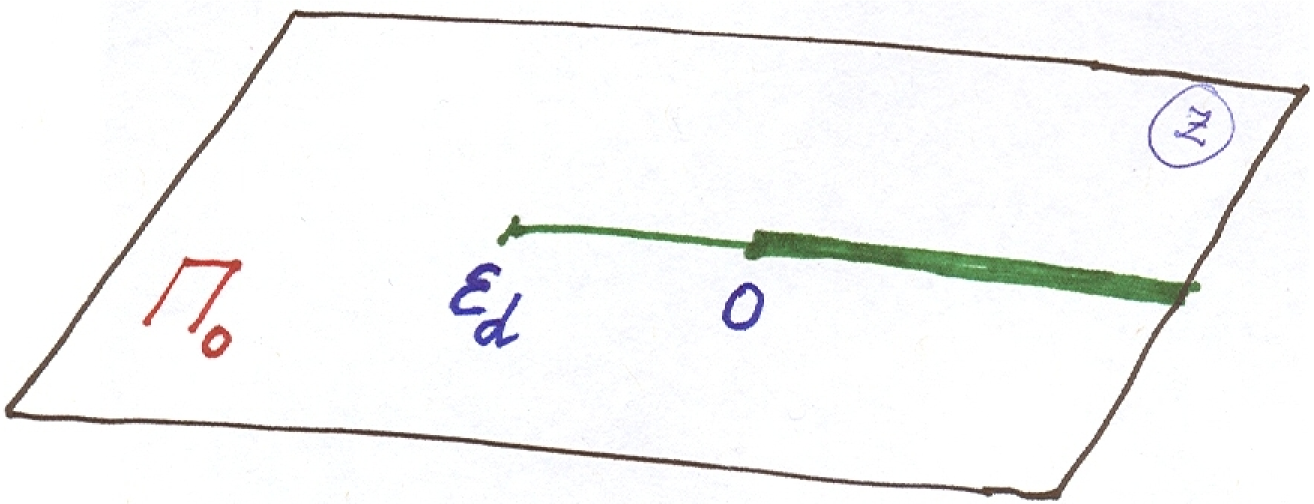
- nnp system
- System of three bosons with nucleon masses
- ^{4}He three-atomic system

6 Conclusions

- Explicit representations for the multi-channel/three-body T-matrix, scattering matrix, and resolvent on unphysical energy sheets not only describe the structure of these quantities but also suggest the ways to calculate multi-channel/three-body resonances.
- A resonance on a sheet Π_l corresponds to a point z on the physical sheet where the truncated scattering matrix $S_l(z)$ has eigenvalue zero,
$$S_l(z)\mathscr{A} = 0.$$
- The corresponding eigenvector $\mathscr{A}$ consists of breakup amplitudes of the resonant state into various channels.

Main references

AM, Phys. Atom. Nucl. **77** (2014), 453
AM, Math. Nachr. **187** (1997), 14
E. A. Kolganova and AM, Phys. Atom. Nucl. **62** (1999), 1179
E. A. Kolganova and AM, Comput. Phys. Comm. **126** (2000), 88



With our computer code we could only calculate the $2 \to 2$ and $2 \to 3$ amplitudes. Hence we were restricted to the study of resonances on the two-cluster unphysical sheet, the one neighboring the physical sheet along the interval $(\varepsilon_d, 0)$.

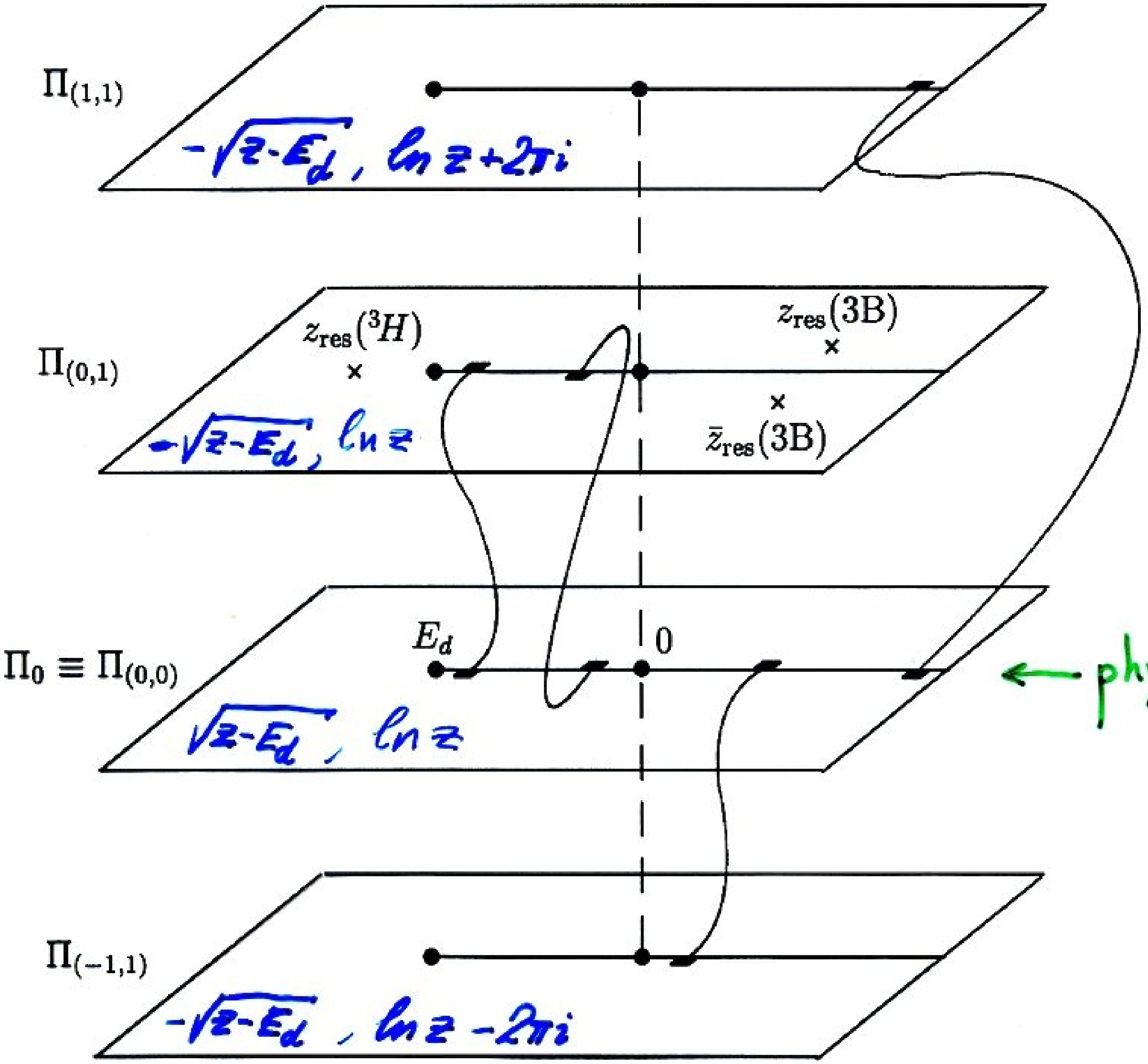
The resonances were looked for as zeros of the scattering matrix

$$S_{(0,1)}(z) = S_0(z) = 1 + 2\mathrm{i}a_0(z),$$

where $a_0(z)$ stands for the s-wave $2 \to 2$ elastic scattering amplitude.

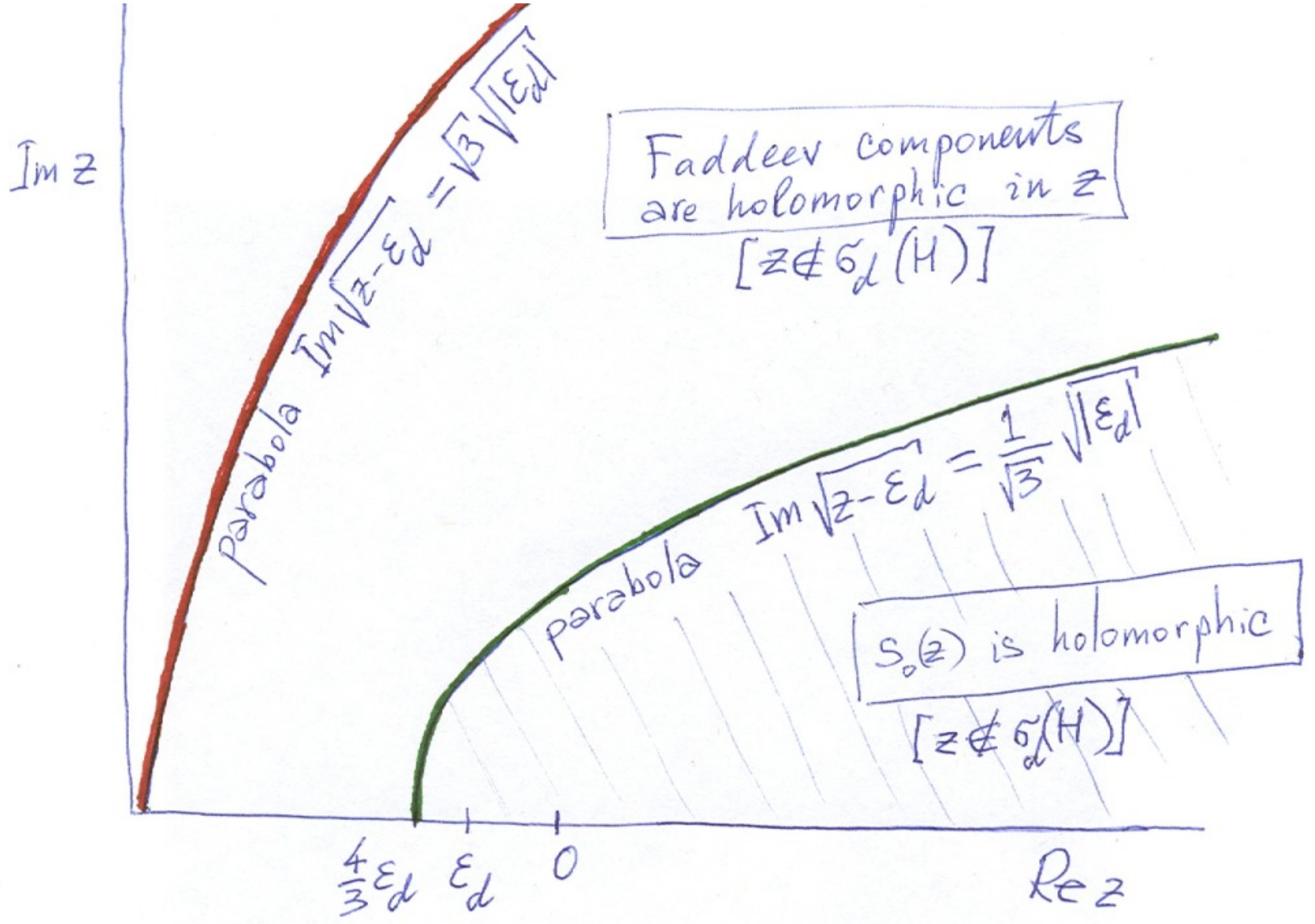
nnp, 3B, $^4He_3$ trimer

$$f(z) = \begin{pmatrix} \ln z \\ (z-E_d)^{1/2} \end{pmatrix}$$



← physical sheet

$\Pi_{(l_0,l_1)}$



Admissible domain in the case of three particles with the same mass; ε_d stands for the deuteron (or dimer) binding energy (the picture is borrowed from [*E. A. Kolganova and AM, Phys. Atom. Nucl.* **62** *(1999), 1179*]; for explanations see this paper).

An advantage: with this approach we can, of course, calculate virtual levels.

More details on formalism

Example of the symmetric ^{4}He$_3$ system. Restrict to a total angular momentum $L=0$. Two-dimensional integro-differential Faddeev equations

$$\left[-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+l(l+1)\left(\frac{1}{x^2}+\frac{1}{y^2}\right)-E\right]\Phi_l(x,y)=\begin{cases}-V(x)\Psi_l(x,y), & x>c\\ 0, & x<c\,.\end{cases} \qquad (6.1)$$

Here, x,y stand for the standard Jacobi variables and c for the core range. The angular momentum l corresponds to a dimer subsystem and a complementary atom; for an S-wave three-boson state. The partial wave function $\Psi_l(x,y)$ is related to the Faddeev components $\Phi_l(x,y)$ by

$$\Psi_l(x,y)=\Phi_l(x,y)+\sum_{l'}\int_{-1}^{+1}d\eta\, h_{ll'}(x,y,\eta)\,\Phi_{l'}(x',y'), \qquad (6.2)$$

where

$$x'=\sqrt{\frac{1}{4}x^2+\frac{3}{4}y^2-\frac{\sqrt{3}}{2}xy\eta}\,, \qquad y'=\sqrt{\frac{3}{4}x^2+\frac{1}{4}y^2+\frac{\sqrt{3}}{2}xy\eta}\,,$$

and $1\le\eta\le 1$.

The functions $\Phi_l(x,y)$ satisfy the boundary conditions

$$\Phi_l(x,y)\,\big|_{x=0} = \Phi_l(x,y)\,\big|_{y=0} = 0\,. \tag{6.3}$$

Moreover, in the hard-core model they are required to satisfy the condition

$$\Phi_l(c,y) + \sum_{l'} \int_{-1}^{+1} d\eta\, h_{ll'}(c,y,\eta)\,\Phi_{l'}(x',y') = 0\,. \tag{6.4}$$

This guarantees the wave function $\Psi_l(x,y)$ to be zero not only at the core boundary $x = c$ but also inside the core domains.

The asymptotic boundary condition for the partial-wave Faddeev components of the two-fragment scattering states reads, as $\rho \to \infty$ and/or $y \to \infty$,

$$\begin{aligned} \Phi_l(x,y;p) \;=\; & \delta_{l0}\psi_d(x)\left\{\sin(py) + \exp(\mathrm{i}py)\left[a_0(p) + o\left(y^{-1/2}\right)\right]\right\} \\ & + \frac{\exp(\mathrm{i}\sqrt{E}\rho)}{\sqrt{\rho}}\left[A_l(\theta) + o\left(\rho^{-1/2}\right)\right]. \end{aligned} \tag{6.5}$$

Here, $\psi_d(x)$ is the dimer wave function, E stands for the scattering energy given by $E = \varepsilon_d + p^2$ with ε_d the dimer energy, and p for the relative momentum conjugate to the variable y. The variables $\rho = \sqrt{x^2+y^2}$ and $\theta = \arctan\dfrac{y}{x}$ are the hyperradius and hyperangle, respectively. The coefficient $a_0(p)$ is nothing but the elastic scattering amplitude, while the functions $A_l(\theta)$ provide us, at $E > 0$, with the corresponding partial-wave Faddeev breakup amplitudes.

$^4He_3$ [*E. A. Kolganova and AM, Phys. Atom. Nucl.* ***62*** *(1999), 1179*]

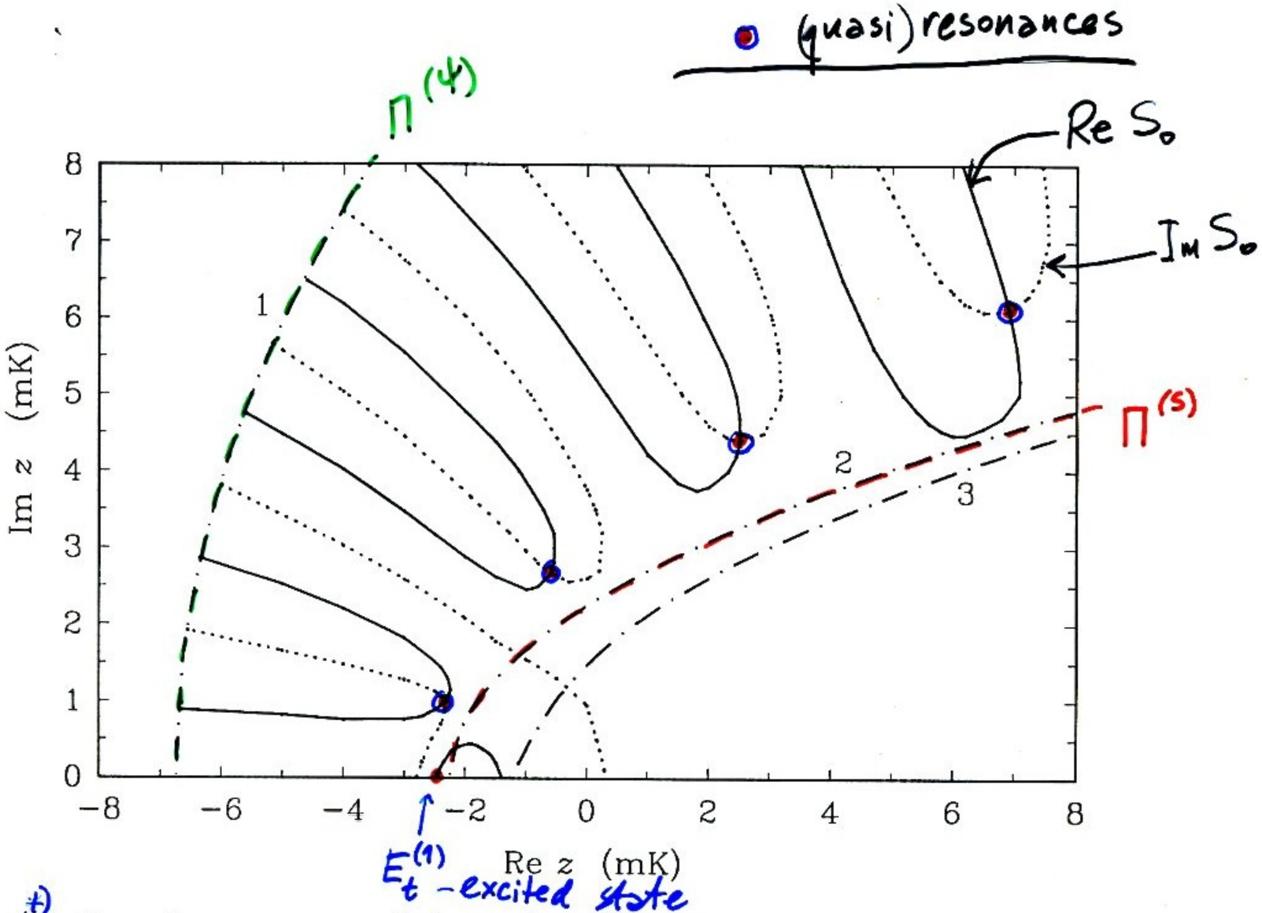


FIG. 1. Root locus curves of the real and imaginary parts of the scattering matrix $S_0(z)$. The solid lines correspond to $\mathrm{Re}\, S_0(z) = 0$ while the tiny dashed lines, to $\mathrm{Im}\, S_0(z) = 0$. The numbers 1, 2, 3 denote the boundaries of the domains $\Pi^{(\Psi)}$, $\Pi^{(S)}$ and $\Pi^{(A)}$, respectively. Complex roots of the function $S_0(z)$ are represented by the crossing points of the curves $\mathrm{Re}\, S_0(z) = 0$ and $\mathrm{Im}\, S_0(z) = 0$ and are located at $(-2.34 + \mathrm{i}\, 0.96)$ mK, $(-0.59 + \mathrm{i}\, 2.67)$ mK, $(2.51 + \mathrm{i}\, 4.34)$ mK and $(6.92 + \mathrm{i}\, 6.10)$ mK.

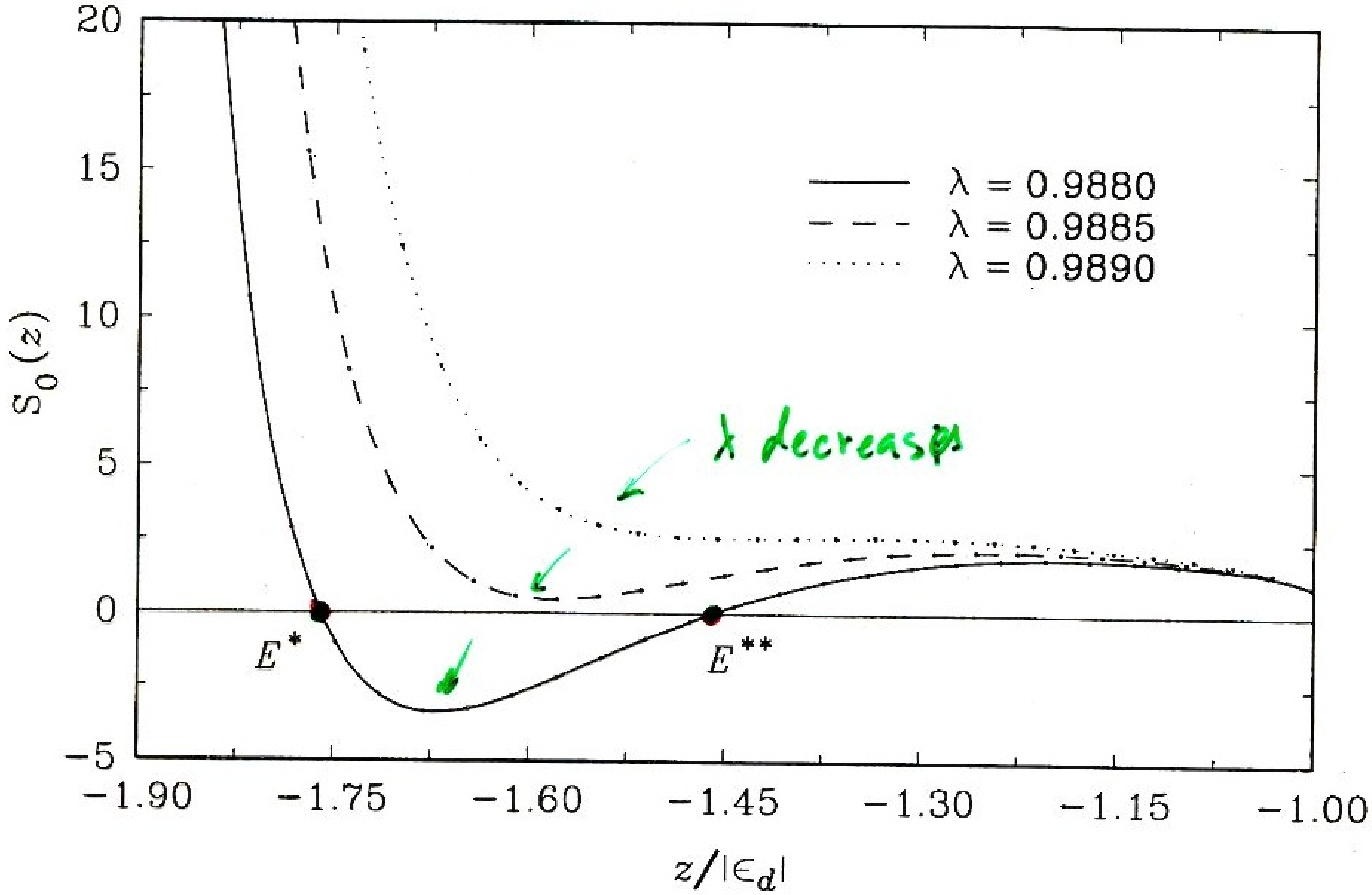
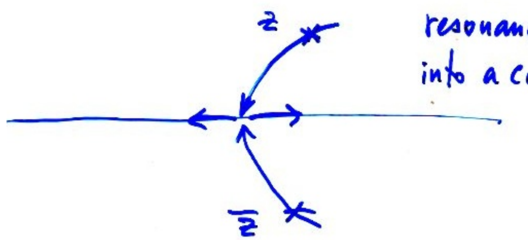


FIG. 2. Graphs of the function $S_0(z)$ at real $z \le \epsilon_d$ for three values of $\lambda < 1$. The notations used: $E^* = E_t^{(2)*}/|\epsilon_d|$, $E^{**} = E_t^{(2)**}/|\epsilon_d|$.



"collision" of conjugate resonances and turning into a couple of virtual levels

The figure below has been borrowed from [E. Kolganova and AM, Proc. of 9th Intern. Conf. on Computational Modelling and Computing in Physics, p. 177]

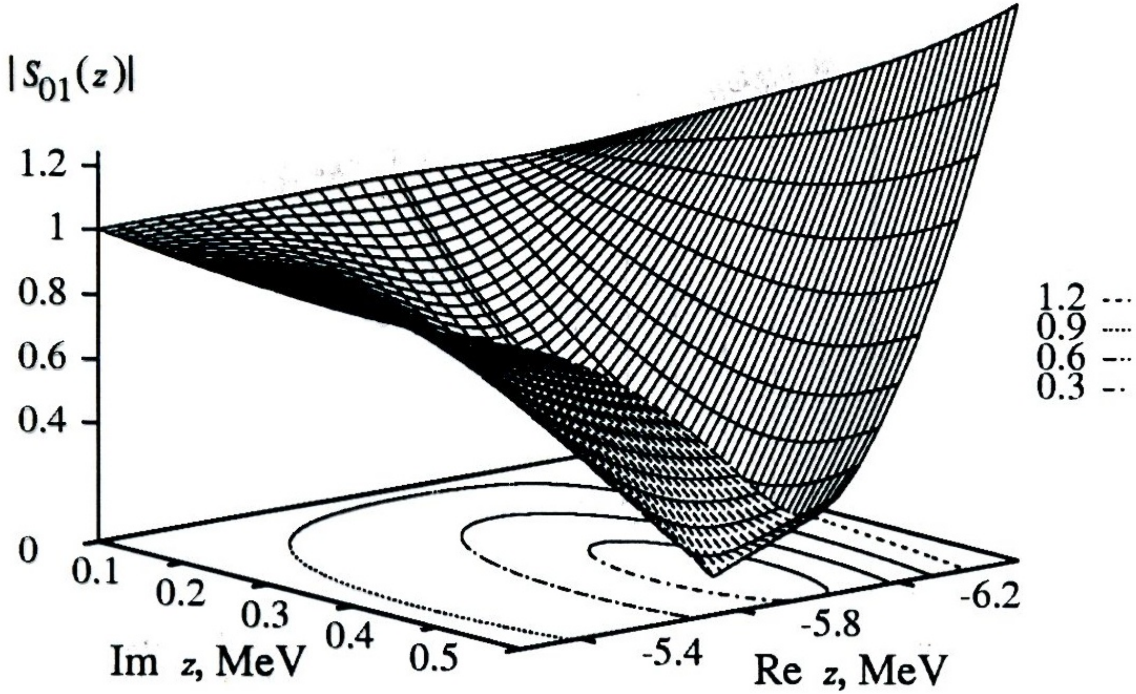
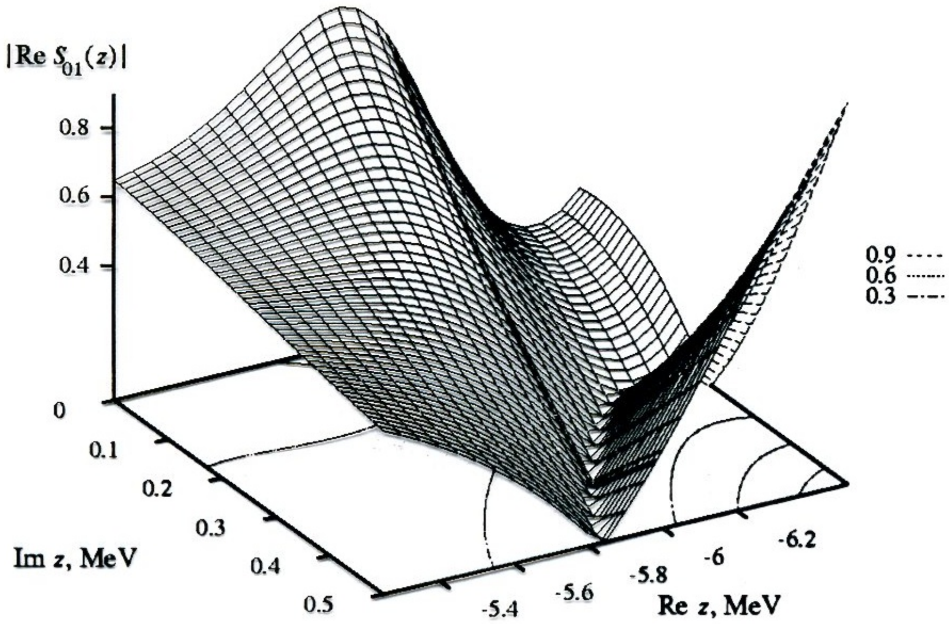
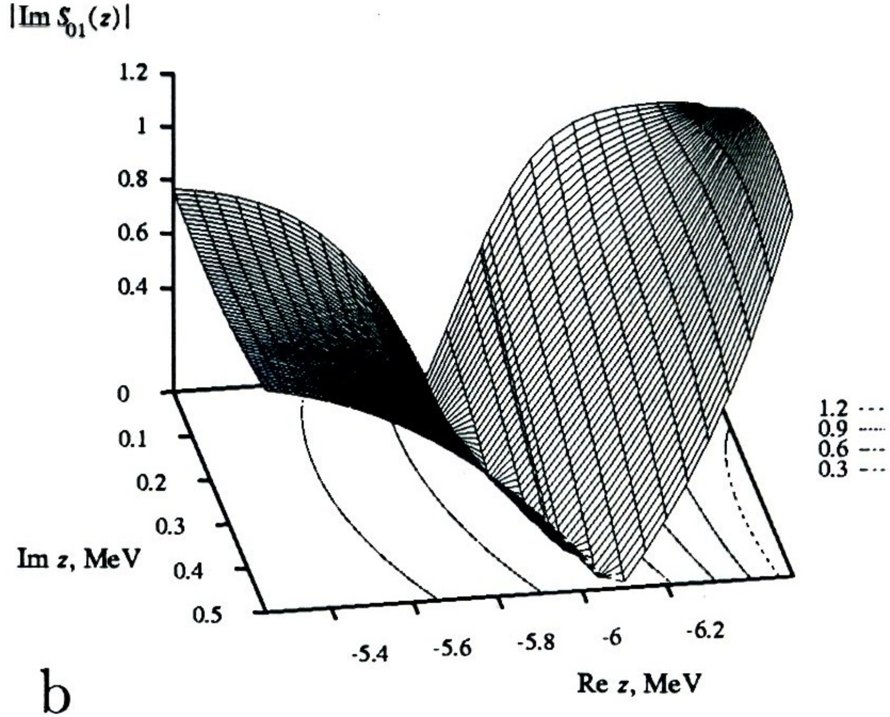


FIG. 2. Surface of the function $|S_{01}(z)|$ in the model system of three bosons with the nucleon masses. The potential $V^G(r)$ is used with the barrier $V_b = 1.5$ MeV. Position of the resonance $z_{\rm res}(3B)$ corresponds to the minimal (zero) value of $|S_{01}(z)|$.



a



b

FIG. 3. Surface of the absolute values of real (a) and imaginary (b) components of the scattering matrix $S_{01}(z)$ in the model system of three bosons with the nucleon masses.