The Algebraic Versions of the Resonating Group Model and the Orthogonality Conditions Model as Fundamentals of Theoretical Approaches to Describing Radiative Capture

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Motivation

Radiative capture reactions $^{A_1}Z_1 + {}^{A_2}Z_2 \rightarrow {}^{(A_1+A_2)}(Z_1+Z_2) + \gamma$

play an important role in nuclear astrophysics.

These reactions are strongly suppressed by the Coulomb barrier

at low energies

Their cross sections are not available

for reliable experimental measurements

Theoretical calculations based on microscopic approaches

are the most justified and promising line of attack

of the low energy cross section problem

Microscopic approach

Physical viewpoint:

- Description of dynamics of all nucleons constituting nuclear system;
- Exact account for Pauli exclusion principle;
- Correct treatment of center of mass motion.

From a mathematical viewpoint, wave functions:

- depend on spin-isospin and space coordinates of all nucleons;
- are fully antisymmetrized for permutations of all pairs of nucleons;
- are translationally invariant.

Nuclear models underlying the existing approaches

- Resonating group model (RGM).
- Fermionic molecular dynamics.
- Variational Monte–Carlo method.
- No-core shell model.
- Direct capture model.
- Potential cluster model.

Algebraic version of the resonating group model (AVRGM)

In single-channel variant of RGM total wave function is sought in form:

$$\Psi^{(1+2)} = \mathrm{A}\left\{\varphi^{(1)}\varphi^{(2)} f(\mathbf{q})\right\}, \qquad \mathbf{q} = \sqrt{\frac{A_1 A_2}{A_1 + A_2}}\left(\mathbf{R}^{(1)}_{\text{c.m.}} - \mathbf{R}^{(2)}_{\text{c.m.}}\right).$$

The main idea of algebraic version (AV) of RGM is expansion of relative motion wave function onto series over the basis of the oscillator functions:

$$f_{\nu lm}(\mathbf{q}) = (-1)^{(\nu-l)/2} N_{\nu l}\, \bar{q}^{\,l} L^{(l+1/2)}_{(\nu-l)/2}(\bar{q}^2)\exp(-\bar{q}^2/2) Y_{lm}(\mathbf{n}_{\mathbf{q}}), \quad N_{\nu l} = \sqrt{\frac{1}{r_0^3}\frac{2\left[(\nu-l)/2\right]!}{\Gamma\left[(\nu+l)/2+3/2\right]}}, \quad \bar{q} = q/r_0.$$

Total wave function can be written as expansion over basis wave functions of AVRGM:

$$\Psi^{(1+2)} = \sum_{J=J_0}^{\infty}\sum_{M=-J}^{J}\sum_{s=|s_1-s_2|}^{s_1+s_2}\sum_{l=|J-s|}^{J+s}\sum_{\nu=\nu_0}^{\infty} C_{J^\pi Mls\nu}\,\Psi^{(1+2)}_{J^\pi Mls\nu}, \qquad \Psi^{(1+2)}_{J^\pi Mls\nu} = \mathrm{A}\left\{\sum_{m+\sigma=M} C^{JM}_{lm\, s\sigma}\left[\varphi^{(1)}_{s_1}\varphi^{(2)}_{s_2}\right]_{s\sigma} f_{\nu lm}(\mathbf{q})\right\}.$$

Expansion coefficients satisfy infinite set of homogeneous linear algebraic equations:

$$\sum_{s=|s_1-s_2|}^{s_1+s_2}\sum_{l=|J-s|}^{J+s}\sum_{\nu=\nu_0}^{\infty}\left(\left\langle J^\pi M\tilde{l}\tilde{s}\tilde{\nu}\middle|H\middle|J^\pi Mls\nu\right\rangle - E\,\delta_{\tilde{s}s}\delta_{\tilde{l}l}\delta_{\tilde{\nu}\nu}\right)C_{J^\pi Mls\nu} = 0,$$

$$\tilde{s} = |s_1 - s_2|,\ldots,s_1+s_2, \qquad \tilde{l} = |J-s|,\ldots,J+s, \qquad \tilde{\nu} = \nu_0, \nu_0+2,\ldots$$

AVRGM equations sets for discrete spectrum and continuum

AVRGM equations set for discrete spectrum is

finite one of homogeneous linear algebraic equations:

$$\sum_{s,\,l}\sum_{\nu=\nu_0}^{\nu_{\max}}\left(\left\langle J^\pi M\tilde{l}\tilde{s}\tilde{\nu}\middle|H\middle|J^\pi Mls\nu\right\rangle - E\,\delta_{\tilde{s}s}\delta_{\tilde{l}l}\delta_{\tilde{\nu}\nu}\right)C^{(\mathrm{D})}_{J^\pi Mls\nu}=0,\qquad \tilde{\nu}=\nu_0,\nu_0+2,\dots,\nu_{\max}.$$

AVRGM equations set for continuum is

finite one of inhomogeneous linear algebraic equations:

$$\sum_{s,\,l}\sum_{\nu=\nu_0}^{\nu_{\mathrm{as}}-2}\left(\left\langle J^\pi M\tilde{l}\tilde{s}\tilde{\nu}\middle|H\middle|J^\pi Mls\nu\right\rangle - E\,\delta_{\tilde{s}s}\delta_{\tilde{l}l}\delta_{\tilde{\nu}\nu}\right)C^{(\mathrm{C})}_{J^\pi Mls\nu}=F_{J^\pi M\tilde{l}\tilde{s}\tilde{\nu}},\qquad \tilde{\nu}=\nu_0,\nu_0+2,\dots,\nu_{\mathrm{as}}.$$

$$F_{J^\pi M\tilde{l}\tilde{s}\tilde{\nu}}=-\sum_{s,\,l}\sum_{\nu=\nu_{\mathrm{as}}}^{\nu_{\max}}\left\langle J^\pi M\tilde{l}\tilde{s}\tilde{\nu}\middle|H\middle|J^\pi Mls\nu\right\rangle C^{(\mathrm{as})}_{J^\pi Mls\nu},$$

$$C^{(\mathrm{as})}_{J^\pi Mls\nu}=\sqrt{\frac{8\pi m(2l+1)r_0}{k^3\hbar\bar{q}_0}}\left[\cos\delta_{Jls}F_l\left(\eta,kr_0\bar{q}_0\right)+\sin\delta_{Jls}G_l\left(\eta,kr_0\bar{q}_0\right)\right],\quad \bar{q}_0=\sqrt{2\nu+3}.$$

Generating Functions Method

$$f_{\nu l m}(\mathbf{q}) = A_{\nu l}\frac{\partial^{\nu}}{\partial R^{\nu}}\int \exp(-q^2/2r_0^2 + \mathbf{qR}/r_0 - R^2/4)Y_{lm}(\mathbf{n_R})d\mathbf{n_R}\Big|_{R=0}$$

$$A_{\nu l} = (-1)^{(\nu-l)/2}\frac{2^{\nu-1/2}}{\pi^{3/2}\nu!}\sqrt{\Gamma\left(\frac{\nu-l+2}{2}\right)\Gamma\left(\frac{\nu+l+3}{2}\right)},$$

R – generating parameter

Generating functions for AVRGM basis:

$$\Phi_{s\sigma}^{(1+2)}(\mathbf{R}) = \mathrm{A}\left\{\left[\varphi_{s_1}^{(1)}\varphi_{s_2}^{(2)}\right]_{s\sigma}\exp(-q^2/2r_0^2 + \mathbf{qR}/r_0 - R^2/4)\right\}$$

Matrix elements:

$$\left\langle J_f^{\pi_f} M_f l_f s_f \nu_f \left| V \right| J_i^{\pi_i} M_i l_i s_i \nu_i \right\rangle = \frac{1}{\kappa_{\nu_f l_f s_f}\kappa_{\nu_i l_i s_i}\nu_f!\nu_i!}\frac{\partial^{\nu_f}}{\partial Q^{\nu_f}}\frac{\partial^{\nu_i}}{\partial R^{\nu_i}} I_{i\to f}(Q,R)\Big|_{R=Q=0}$$

$$I_{i\to f}(Q,R) = \sum_{m_f\sigma_f m_i\sigma_i} C_{l_f m_f\, s_f\sigma_f}^{J_f M_f} C_{l_i m_i\, s_i\sigma_i}^{J_i M_i}\iint Y_{l_f m_f}^{*}(\mathbf{n_Q})\left\langle \mathbf{Q}, s_f\sigma_f \left| V \right| \mathbf{R}, s_i\sigma_i\right\rangle Y_{l_i m_i}(\mathbf{n_R})d\mathbf{n_Q}d\mathbf{n_R}$$

$$(\nu!)^2\kappa_{\nu l s}^2 = \frac{\partial^{\nu}}{\partial Q^{\nu}}\frac{\partial^{\nu}}{\partial R^{\nu}}\iint Y_{lm}^{*}(\mathbf{n_Q})\langle \mathbf{Q}, s\sigma | \mathbf{R}, s\sigma\rangle Y_{lm}(\mathbf{n_R})d\mathbf{n_Q}d\mathbf{n_R}\Big|_{R=Q=0}$$

The algebraic version of the orthogonality conditions model (AVOCM)

In many cases wave functions of initial and final states can be expressed by Slater's determinants or their sum:

$$|i\rangle = \mathrm{A}\prod_{j=1}^{A}\phi_j(j) = \frac{1}{\sqrt{A!}}\sum_{\{j_1,j_2,\ldots,j_A\}}(-1)^{P(\{j_1,j_2,\ldots,j_A\})}\phi_{j_1}(1)\phi_{j_2}(2)\ldots\phi_{j_A}(A),$$

$$|f\rangle = \mathrm{A}\prod_{j=1}^{A}\varphi_j(j) = \frac{1}{\sqrt{A!}}\sum_{\{j_1,j_2,\ldots,j_A\}}(-1)^{P(\{j_1,j_2,\ldots,j_A\})}\varphi_{j_1}(1)\varphi_{j_2}(2)\ldots\varphi_{j_A}(A).$$

If single-particle states satisfy orthogonality condition $\langle\varphi_m|\phi_n\rangle \sim \delta_{mn}$,

then matrix elements of an operator $V = \sum_{k>j=1}^{A} V(k,j)$ are written in the form:

$$\langle f|V|i\rangle = \sum_{m>n=1}^{A}\Big(\langle\varphi_m(1)|\langle\varphi_n(2)|V(1,2)|\phi_n(2)\rangle|\phi_m(1)\rangle - \langle\varphi_m(1)|\langle\varphi_n(2)|V(1,2)|\phi_m(2)\rangle|\phi_n(1)\rangle\Big)\prod_{l\neq m,n}\langle\varphi_l|\phi_l\rangle.$$

exchange terms

Neglecting exchange effects caused by Pauli exclusion principle in interaction between nucleons composing two different clusters leads to the algebraic version of the orthogonality conditions model (AVOCM).

Hamiltonian and Nuclear Potential

Hamiltonian of the system: $H = T - T_{\text{c.m.}} + V_{\text{coul}} + V_{\text{nucl}}^{(\text{H-N})}$

$T - T_{\text{c.m.}} = -\dfrac{\hbar^2}{2mA}\sum_{i>j=1}^{A}\left(\nabla_i - \nabla_j\right)^2$ – kinetic energy

$V_{\text{coul}} = \sum_{i>j=1}^{Z}\dfrac{e^2}{r_{ij}}$ – Coulomb interaction of the protons

Modified Hasegawa–Nagata potential:

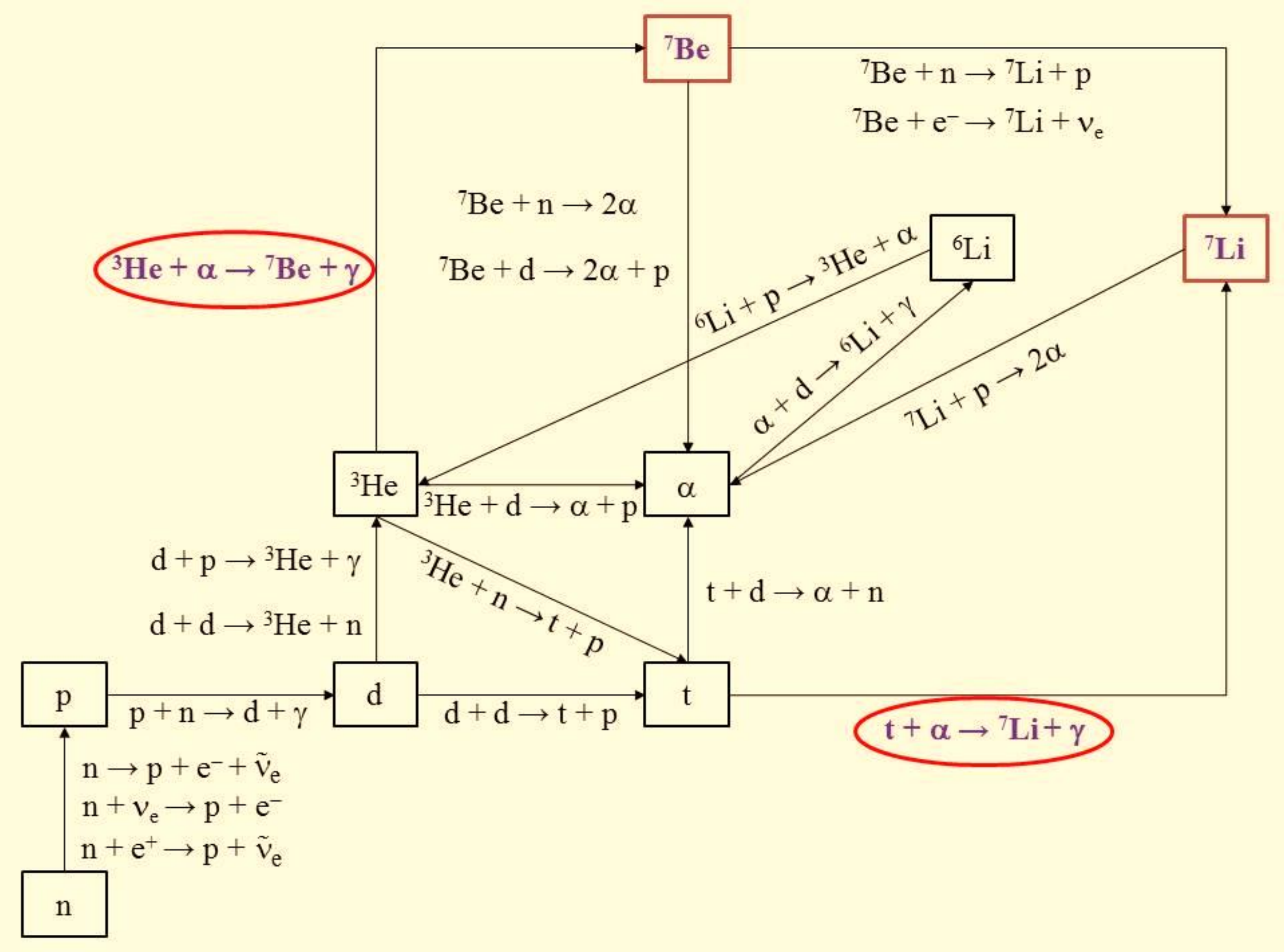
$$V_{\text{nucl}}^{(\text{H-N})} = \sum_{i>j=1}^{A}\left(V_{ij}^{(\text{c})} + V_{ij}^{(ls)} + V_{ij}^{(\text{t})}\right)$$

$$V_{ij}^{(\text{c})} = \sum_{n=1}^{3} V_n^{(\text{c})}\left(w_n^{(\text{c})} + (1-g_c)m_n^{(\text{c})} - g_c m_n^{(\text{c})}P_{ij}^{\sigma}P_{ij}^{\tau} + b_n^{(\text{c})}P_{ij}^{\sigma} - h_n^{(\text{c})}P_{ij}^{\tau}\right)\exp(-\mu_n^{(\text{c})}r_{ij}^2)$$

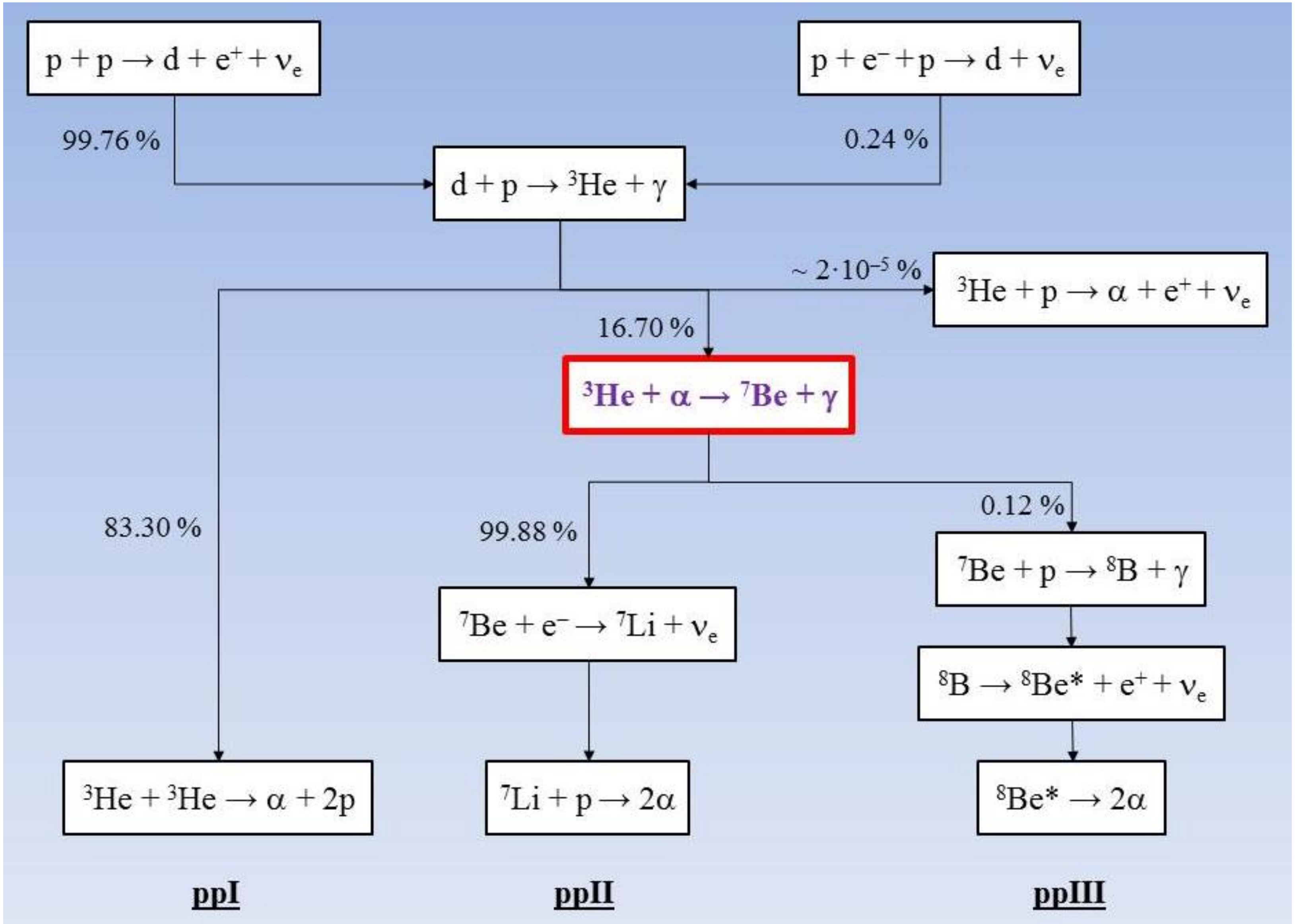
$$V_{ij}^{(ls)} = \sum_{n=1}^{2}\frac{V_n^{(ls)}}{2}\left(w_n^{(ls)} - h_n^{(ls)}P_{ij}^{\tau}\right)\left[\mathbf{r}_{ij}\times(\mathbf{p}_i - \mathbf{p}_j)\right](\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_j)\exp(-\mu_n^{(ls)}r_{ij}^2)$$

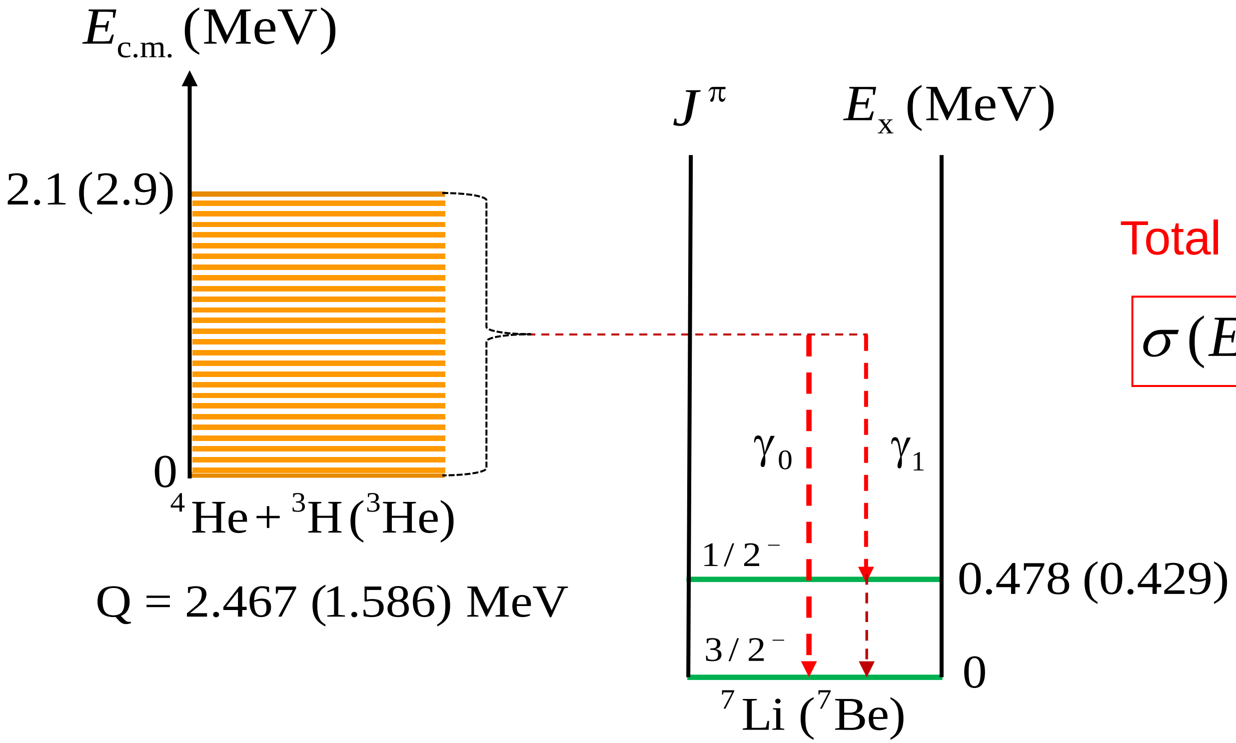
$$V_{ij}^{(\text{t})} = \sum_{n=1}^{3} V_n^{(\text{t})}\left(w_n^{(\text{t})} - h_n^{(\text{t})}P_{ij}^{\tau}\right)\left(3(\boldsymbol{\sigma}_i\mathbf{r}_{ij})(\boldsymbol{\sigma}_j\mathbf{r}_{ij}) - (\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j)\mathbf{r}_{ij}^2\right)\exp(-\mu_n^{(\text{t})}r_{ij}^2)$$

The basic reactions of primordial nucleosynthesis



pp cycle of hydrogen burning





Total cross section:

$$\sigma(E_{\mathrm{c.m.}}) = \sigma_0(E_{\mathrm{c.m.}}) + \sigma_1(E_{\mathrm{c.m.}})$$

Partial cross section:

$$\sigma_{i\to f}(E_{\mathrm{c.m.}}) = \frac{8\pi}{9\hbar(2l_i+1)}\left(\frac{E_\gamma}{\hbar c}\right)^3 \left|\sum_{\nu_i,\nu_f} C^{(\mathrm{D})}_{f\nu_f}\left\langle f\nu_f \left\| M_1^{\mathrm{E}} \right\| i\nu_i \right\rangle C^{(\mathrm{C})}_{i\nu_i}\right|^2, \quad f=(J_f^{\pi_f} l_f s),\ i=(J_i^{\pi_i} l_i s),\ s=1/2$$

In case of ^7Li (^7Be) formed in the ground state: $\left(J_f^{\pi_f}, l_f\right) = \left(3/2^-, 1\right)$, $\left(J_i^{\pi_i}, l_i\right) = \left(1/2^+, 0\right), \left(3/2^+, 2\right), \left(5/2^+, 2\right)$.

In case of ^7Li (^7Be) formed in the first excited state: $\left(J_f^{\pi_f}, l_f\right) = \left(1/2^-, 1\right)$, $\left(J_i^{\pi_i}, l_i\right) = \left(1/2^+, 0\right), \left(3/2^+, 2\right)$.

Approaches to the following calculations

Cross section in terms of astrophysical S-factor: $\sigma(E_{\text{c.m.}}) = \frac{1}{E_{\text{c.m.}}} \exp\left(-\sqrt{E_{\text{G}} / E_{\text{c.m.}}}\right) S(E_{\text{c.m.}})$

$$\Longrightarrow \quad S(E_{\text{c.m.}}) = E_{\text{c.m.}} \exp\left(\sqrt{E_{\text{G}} / E_{\text{c.m.}}}\right) \sigma(E_{\text{c.m.}}), \qquad E_{\text{G}} = 2mc^2 \left(\frac{\pi e^2 Z_1 Z_2}{\hbar c}\right)^2 \frac{A_1 A_2}{A_1 + A_2}$$

E_{G} = 6.76 (27.04) MeV – Gamow energy for the α + t (α + h) system.

$$\sigma_{i \to f}(E_{\text{c.m.}}) = \frac{8\pi}{9\hbar(2l_i + 1)} \left(\frac{E_\gamma}{\hbar c}\right)^3 \left| \sum_{\nu_i, \nu_f} C^{(\text{D})}_{f\nu_f} \left\langle f\nu_f \left\| M_1^{\text{E}} \right\| i\nu_i \right\rangle C^{(\text{C})}_{i\nu_i} \right|^2, \quad f = (J_f^{\pi_f} l_f s), \; i = (J_i^{\pi_i} l_i s), \; s = 1/2$$

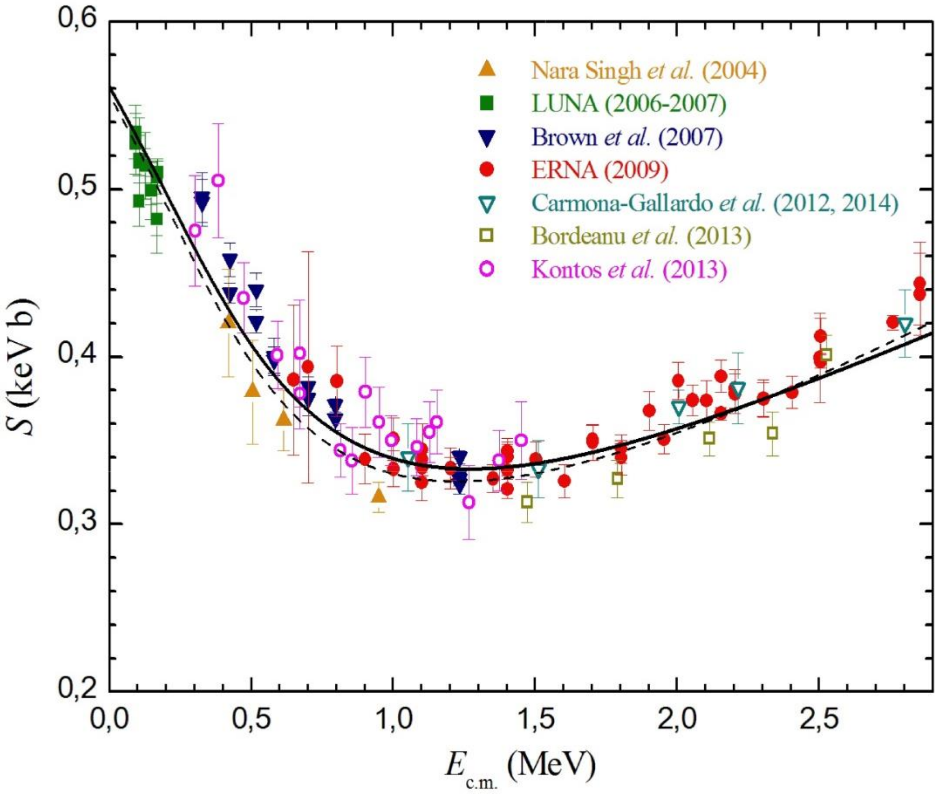
Unknown quantities: $C^{(\text{D})}_{f\nu_f}, \; C^{(\text{C})}_{i\nu_i}, \; \left\langle f\nu_f \left\| M_1^{\text{E}} \right\| i\nu_i \right\rangle.$

Approaches are based on:

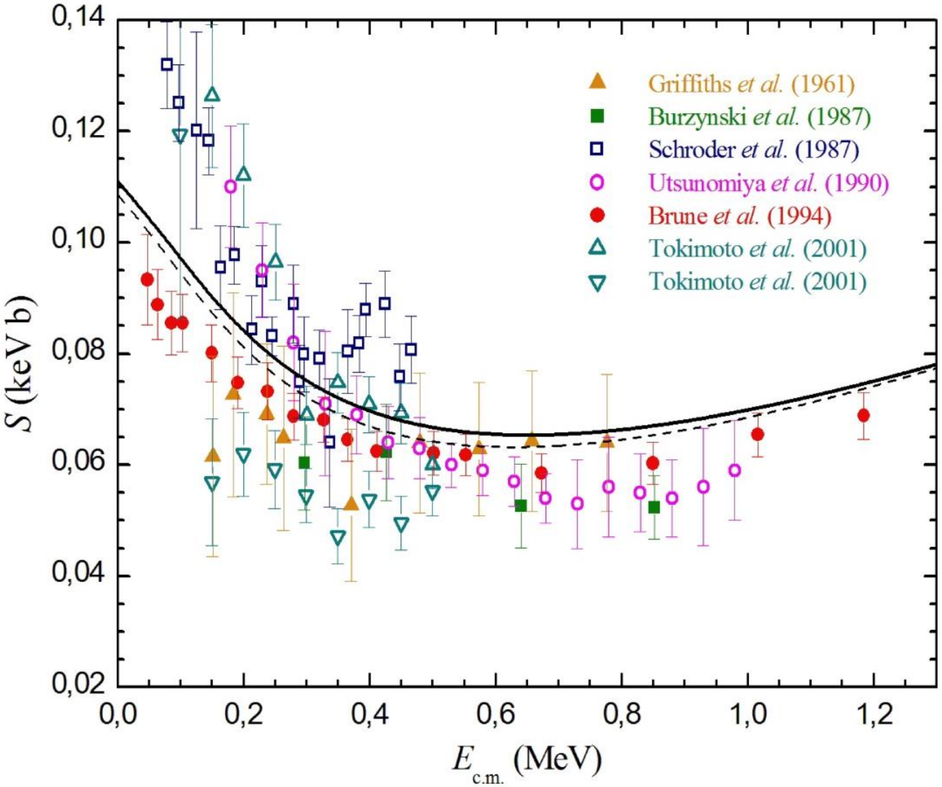
- Single-scale AVRGM (usual AVRGM approach);
- Single-scale AVRGM + AVOCM (combined approach);
- Multi-scale AVRGM (extended AVRGM approach).

Astrophysical S-factors for the $^3He(\alpha,\gamma)^7Be$ and $^3H(\alpha,\gamma)^7Li$ reactions

$^3He(\alpha,\gamma)^7Be$



$^3H(\alpha,\gamma)^7Li$



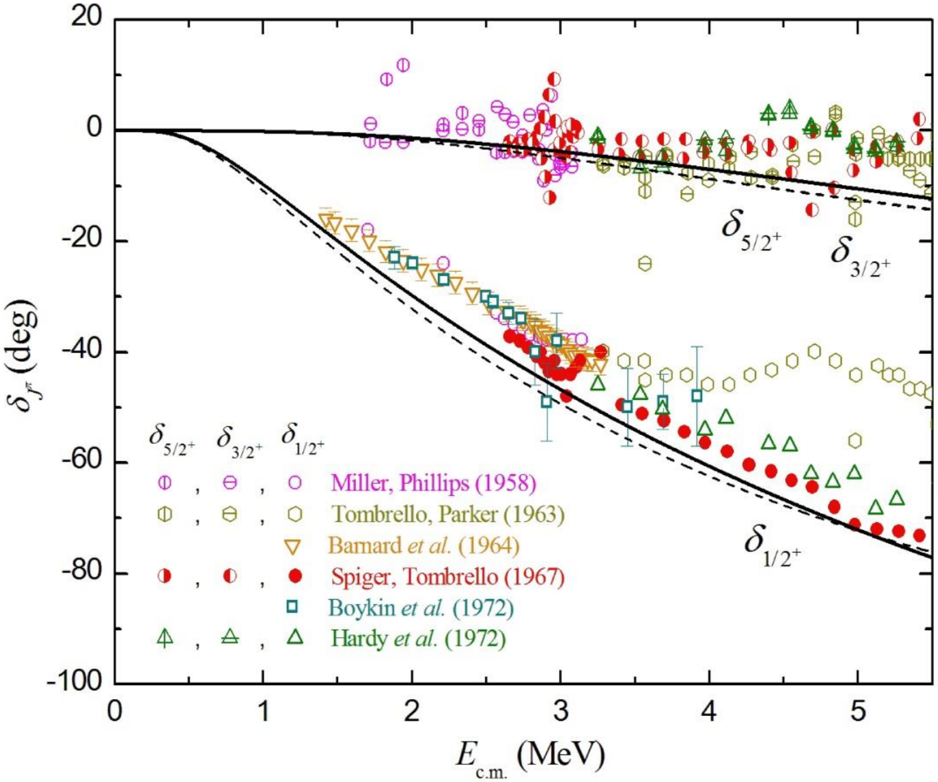
The solid lines are calculation within usual AVRGM approach;

The dashed lines are calculation within approach combining AVRGM and AVOCM.

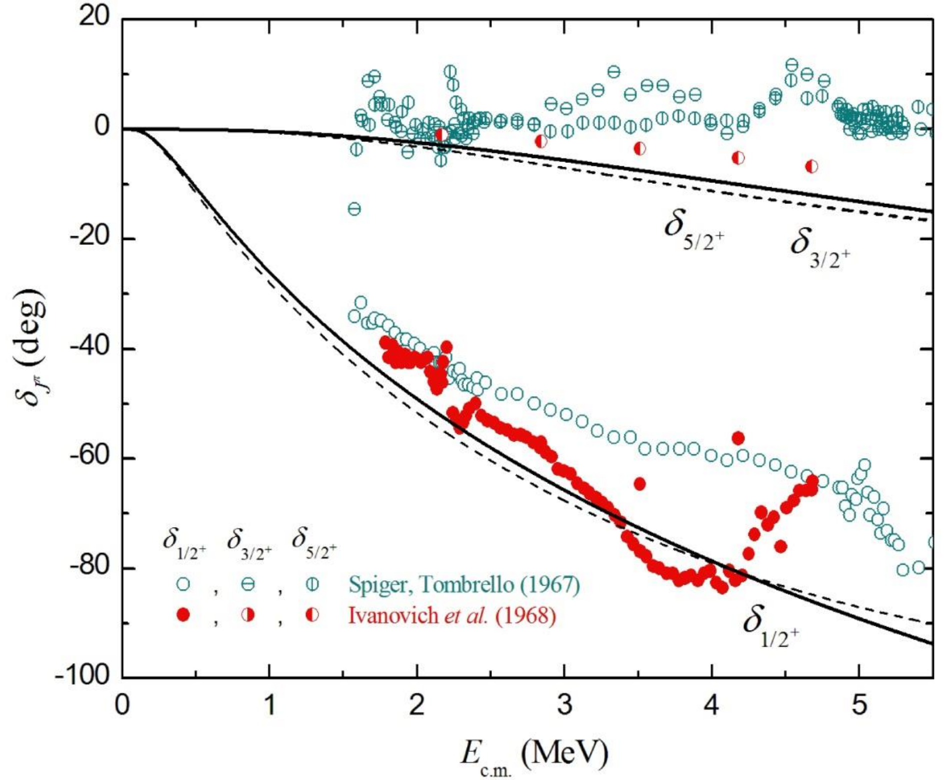
$$r_0 = 1.22 \text{ fm}, \quad g_c = 1.035$$

Scattering phase shifts for $^4He + {^3He}$ and $^4He + {^3H}$ systems

$^4He + {^3He}$



$^4He + {^3H}$



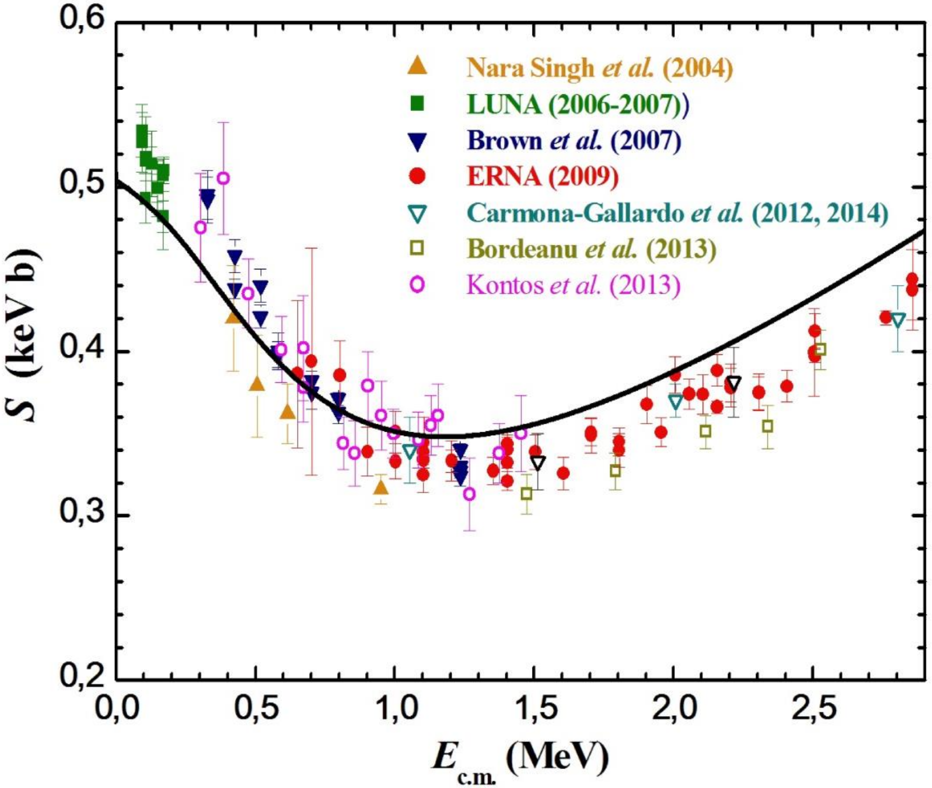
The solid lines are calculation in usual AVRGM approach;

The dashed lines are calculation in combined approach.

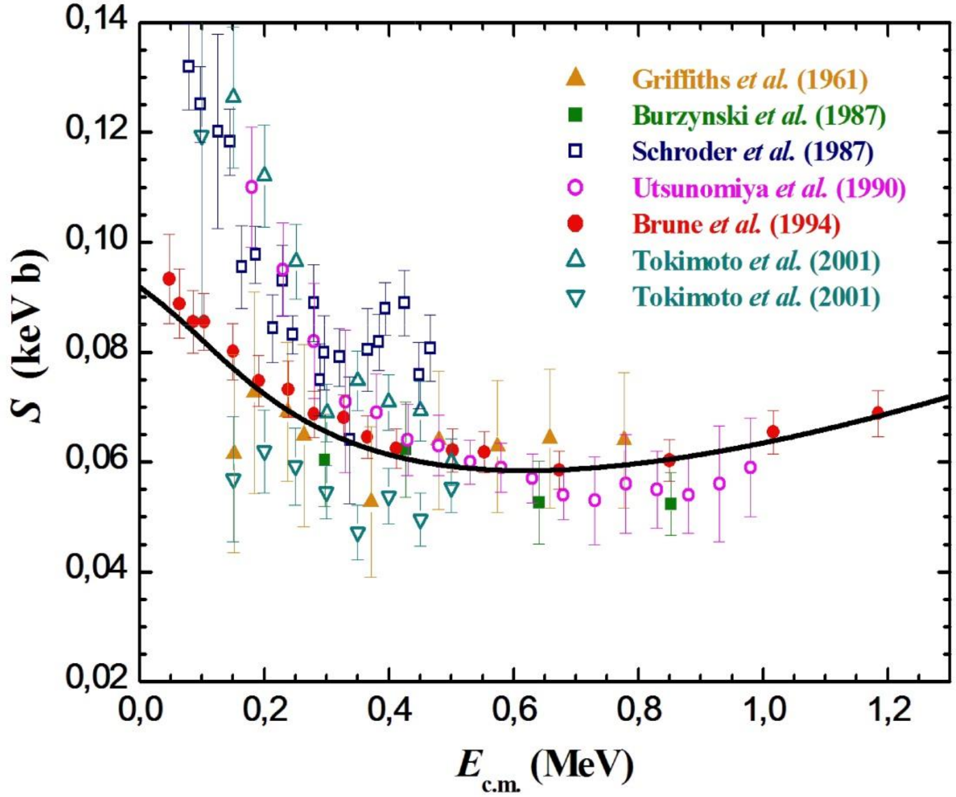
$$r_0 = 1.22 \text{ fm}, \ g_c = 1.035$$

Astrophysical S-factors for the $^3He(\alpha,\gamma)^7Be$ and $^3H(\alpha,\gamma)^7Li$ reactions in the framework of the extended AVRGM approach

$^3He(\alpha,\gamma)^7Be$



$^3H(\alpha,\gamma)^7Li$

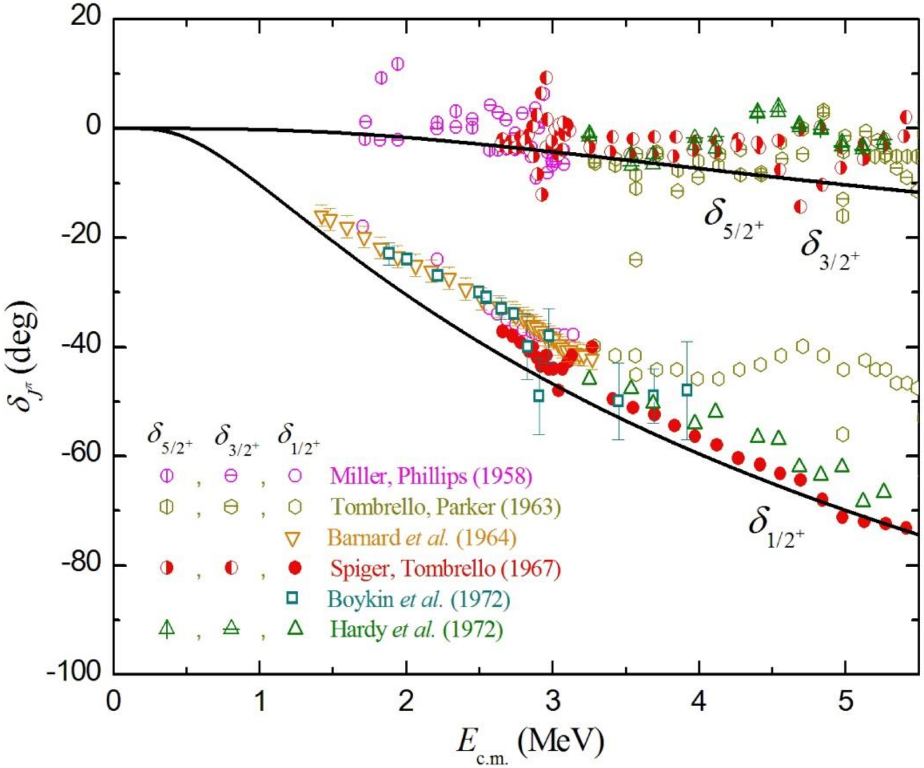


r_{020} = 1.307 fm, r_{021} = 1.421 fm; r_{020} = 1.303 fm, r_{021} = 1.282 fm;

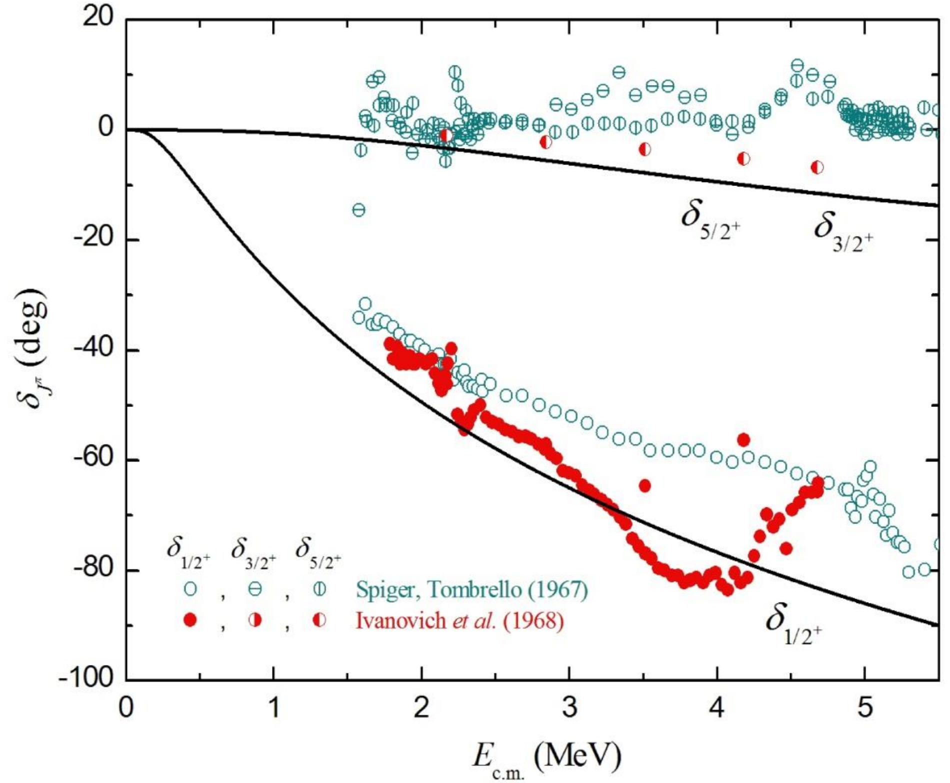
r_{01} = 1.386 fm, g_c = 0.977.

Scattering phase shifts for ^{4}He + ^{3}He and ^{4}He + ^{3}H systems within extended AVRGM approach

^{4}He + ^{3}He



^{4}He + ^{3}H



$r_{01} = 1.386$ fm, $g_c = 0.977$.

Energies for the considered nuclei (extended AVRGM approach)

Energy (MeV)	Experiment	Calculation
$E(^4He)$	−28.296	−28.296
$E(^3H)$	−8.482	−6.467
$E(^3He)$	−7.718	−5.638
$E(^7Li)$	−39.244	−37.230
$E(^7Li^*)$	−38.766	−36.752
$\varepsilon_0^{(\alpha+t)}$	2.467	2.467
$\varepsilon_1^{(\alpha+t)}$	1.989	1.989
$E(^7Be)$	−37.600	−35.520
$E(^7Be^*)$	−37.171	−35.091
$\varepsilon_0^{(\alpha+h)}$	1.586	1.586
$\varepsilon_1^{(\alpha+h)}$	1.157	1.157

^7Li rows ($E(^7Li)$ to $\varepsilon_1^{(\alpha+t)}$): $r_{020} = 1.303$ fm, $r_{021} = 1.282$ fm;

^7Be rows ($E(^7Be)$ to $\varepsilon_1^{(\alpha+h)}$): $r_{020} = 1.307$ fm, $r_{021} = 1.421$ fm;

All rows: $r_{01} = 1.386$ fm, $g_c = 0.977$.

$E(^7Li)$, $E(^7Li^*)$, $E(^7Be)$, $E(^7Be^*)$ – energies of the ground and first excited states of ^7Li and ^7Be nuclei;

$E(^4He)$, $E(^3He)$, $E(^3H)$ – energies of the ground states of ^4He, ^3He and ^3H nuclei.

Conclusion

- Theoretical approaches based on AVRGM and AVOCM have been developed.

- The mirror radiative capture reactions $^{3}H(\alpha,\gamma)^{7}Li$ and $^{3}He(\alpha,\gamma)^{7}Be$ have been studied in the framework of these approaches.

- All the approaches demonstrate opportunities to describe the energy dependences for the astrophysical S-factors of the considered reactions and for the phase shifts of scattering of the nuclei in the entrance channels adequately.

Conclusion

- Extended AVRGM approach compared to the others allows to achieve the unified description of the energy dependences of the astrophysical S-factors for the radiative capture and the phase shifts for the scattering in both the ^{4}He + ^{3}H and ^{4}He + ^{3}He systems along with the exact breakup threshold positions for the formed ^{7}Li and ^{7}Be nuclei.

- It is shown that the exchange terms of interaction between the nuclei turn out to be insignificant in the considered energy ranges.

Thank you for attention

Electric dipole operator matrix elements for seven-nucleon ^{4}He + ^{3}H and ^{4}He + ^{3}He systems

Electric dipole operator: $M^{\mathrm{E}}_{1\mu} = e\sum_{i=1}^{A} g_L(i)\left|\mathbf{r}_i - \mathbf{R}_{\mathrm{c.m.}}\right| Y_{1\mu}(\mathbf{n}_{\mathbf{r}_i-\mathbf{R}_{\mathrm{c.m.}}}), \quad g_L(i) = \frac{1}{2} - t_{3,i}.$

Reduced matrix elements of the electric dipole operator between AVRGM basis wave functions:

$$\left\langle J_f^{\pi_f} l_f s \nu_f, r_{02} \left\| M_1^{\mathrm{E}} \right\| J_i^{\pi_i} l_i s \nu_i, r_{01} \right\rangle = \varsigma(-1)^{J_i+l_f+s+1} \frac{e}{14}\left(\frac{2r_{01}r_{02}}{r_{01}^2+r_{02}^2}\right)^{10} \sqrt{\frac{3(2l_i+1)(2J_i+1)}{\pi}} \begin{Bmatrix} l_i & s & J_i \\ J_f & 1 & l_f \end{Bmatrix} \times$$

$$\times \frac{C^{l_f 0}_{l_i 0 10} C^{J_f M_f}_{J_i M_i 1\mu}}{\kappa_{\nu_f l_f s}\kappa_{\nu_i l_i s}\nu_f!\nu_i!}\left(r_{01}\nu_f \frac{\partial^{\nu_f-1}}{\partial Q^{\nu_f}} \frac{\partial^{\nu_i}}{\partial R^{\nu_i}} U_{l_i}(Q, r_{02}; R, r_{01}) + r_{02}\nu_i \frac{\partial^{\nu_f}}{\partial Q^{\nu_f}} \frac{\partial^{\nu_i-1}}{\partial R^{\nu_i}} U_{l_f}(Q, r_{02}; R, r_{01}) \right)\Bigg|_{Q=R=0},$$

$$U_l(Q, r_{02}; R, r_{01}) = 2\pi \int_{-1}^{1} U(\mathbf{Q}, r_{02}; \mathbf{R}, r_{01}; t) P_l(t)\,dt,$$

$$U(\mathbf{Q}, r_{02}; \mathbf{R}, r_{01}; t) = \exp\left(\frac{3(r_{01}^2 - r_{02}^2)(Q^2 - R^2) - 9r_{01}r_{02}QRt}{7(r_{01}^2 + r_{02}^2)}\right)\left[\exp\left(\frac{r_{01}r_{02}}{r_{01}^2+r_{02}^2}QRt\right) - 1\right]^3,$$

$$\kappa^2_{\nu l s} = \frac{2\pi}{\nu!}\left[\left(\frac{6}{7}\right)^{\nu} - 3\left(\frac{5}{14}\right)^{\nu} + 3\left(-\frac{1}{7}\right)^{\nu} - \left(-\frac{9}{14}\right)^{\nu}\right]\varepsilon_{\nu l}, \qquad \Pi_{J_i J_f l_i} = \sqrt{(2J_i+1)(2J_f+1)(2l_i+1)},$$

$$\varsigma = \begin{cases} -1, & \text{for } ^4\text{He} + {}^3\text{H}; \\ 1, & \text{for } ^4\text{He} + {}^3\text{He}. \end{cases} \qquad \varepsilon_{\nu l} = \begin{cases} \dfrac{2^{l+1}\nu!\left[(\nu+l)/2\right]!}{\left[\nu+l+1\right]!\left[(\nu-l)/2\right]!}, & l \le \nu,\ l+\nu - \text{even}; \\ 0, & \text{in other cases.} \end{cases}$$